AFGL-TR-81-0106(II)

AD A098042

LEVEL III

Grant number: AFOSR – 77 – 3224

TROPOSPHERIC – STRATOSPHERIC TIDAL INVESTIGATIONS. Part II.

THE VERTICAL STRUCTURE OF ATMOSPHERIC OSCILLATIONS FORMULATED BY CLASSICAL TIDAL THEORY.

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28 February 1981

Final Scientific Report, ~~Part II (of 2 parts)~~
1 December 1976 – 30 April 1981

Approved for public release; distribution unlimited.

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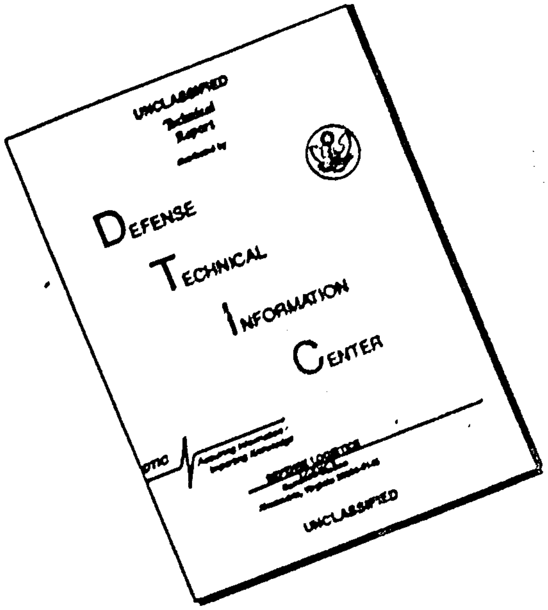

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AFGL-TR-81-0106 (II)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. Report Number Final , Part II (of 2 parts)	**2. Govt Accession No.**	**3. Recipient's Catalog Number**
4. Title (and Subtitle) TROPOSPHERIC-STRATOSPHERIC INVESTIGATIONS, The vertical structure of atmospheric oscillations formulated by classical tidal theory		**5. Type of Report & Period Covered** Final Scientific Report 1 December 1976 - 30 April 1981 **6. Performing Org. Report Number**
7. Author(s) G. V. Groves		**8. Contract or Grant Number** AFOSR - 77 - 3224
9. Performing Organization Name and Address Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, England.		**10. Program Element, Project, Task Area & Work Unit Numbers**
11. Controlling Office Name and Address Air Force Geophysics Laboratory, L. G. Hanscom Field, Bedford, Massachusetts, 01730, USA.		**12. Report Date** 28 February 1981 **13. Number of Pages** 83
14. Monitoring Agency Name and Address European Office of Aerospace Research and Development, Box 14, FPO, New York 09510, USA.		**15.** Unclassified
16. & 17. Distribution Statement **Approved for public release; distribution unlimited.**		
18. Supplementary Notes		
19. Key Words Atmosphere, oscillations, tides		

20. Abstract

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FORM 1473

The vertical structure of atmospheric oscillations formulated by classical tidal theory

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Abstract

From the equations of classical tidal theory with Newtonian cooling (Chapman & Lindzen, 1970), formulae are obtained for wind, temperature and pressure oscillations generated by thermal, gravitational and lower-boundary excitations of given frequency. The analysis is an extension of that of Butler & Small (1963) who formulated solutions of the vertical structure equation in terms of two independent solutions of the homogeneous equation and derived expressions for surface pressure oscillations. A comprehensive formulation is presented for wind, temperature and pressure oscillations as a function of height with the above-mentioned sources of excitation and

an upper-boundary radiation condition. The formulae obtained are applied at the surface leading to evaluations of the surface oscillation weighting function $W_p(z)$ which weights the thermal excitation at height z according to its differential contribution to the surface oscillation. The formulae are shown to simplify at heights above a region of excitation and evaluations are undertaken of the thermal response weighting function $W_t(z)$ which weights the thermal excitation at height z according to its differential contribution to the oscillation at any height above the region of thermal excitation. Computational procedures are described for obtaining two independent solutions of the homogeneous equation and results are presented for an adopted profile of atmospheric scale height. The problem of deriving the surface pressure oscillation due to a tidal potential is briefly reviewed and results are presented as an example of the application of formulae that have been derived.

CONTENTS

1. Introduction

Since the time of Laplace periodic oscillations of the atmosphere as a whole have provided a subject of constant dynamical research. Tidal components, on account of their known periodicities, have formed a major part of this study; and reviews of investigations before 1950 have been presented by Wilkes (1949) and Chapman (1951). Interest up to that time centred chiefly around the periods of free atmospheric oscillation and the so-called resonance theory, which attempted unsuccessfully to account for the relatively large magnitude of the solar semi-diurnal barometric oscillation in terms of a free period of very close to 12 solar hours. Subsequently, the thermal excitation of tides has received increased attention and was included in the review of Chapman & Lindzen (1970) from which the equations of classical tidal theory have been taken for the developments of this paper. In this Introduction some of the main features and developments of classical tidal theory are summarized.

The theory of oscillations in a compressible atmosphere was initially developed as an extention of that of Laplace and others (Hough, 1898) for a liquid ocean of uniform depth. Whereas for an ocean the velocity and pressure variations were independent of depth, the atmospheric problem introduced a dependence on height z which under simplifying assumptions could be expressed in terms of the independent variable

$$x = -\ln p_o/p_{oo} = \int_0^z dS/H(S) \qquad (1.1)$$

where p_o is unperturbed pressure, p_{oo} is surface pressure and H is atmospheric scale height. The fundamental equation was

$$\frac{d^2y}{dx^2} + \left[-\frac{1}{4} + \frac{1}{h}\left(\frac{\gamma-1}{\gamma}H + \frac{dH}{dx}\right)\right]y = 0 \qquad (1.2)$$

where γ is the ratio of specific heats of air and $ye^{\frac{1}{2}x}$ is velocity divergence.

The quantity h in (1.2) is a constant of separation between vertically- and horizontally-dependent terms in the equations of motion and appears also in the equation for the latitudinal variation of the pressure perturbation at a given height. This variation is identical with that of the perturbed depth of an ocean of otherwise uniform depth h (Laplace's tidal equation), and it is customary to refer to h as an equivalent depth of the atmosphere. The choice of h in (1.2) is limited to values $h_n^{\sigma,s}$ which are the eigenvalues of Laplace's tidal equation for a given angular frequency of oscillation σ and zonal wave number s.

Use of the alternative forms of wave representation $e^{i(s\phi+\sigma t)}$ and $e^{i(s\phi-\sigma t)}$, where ϕ is longitude and t is time, has previously been discussed (Groves, 1979): the form that appears more frequently in the literature is $e^{i(s\phi+\sigma t)}$, σ being positive and s = 0, ±1, ±2, . . so that waves travel westward if $s > 0$ and eastward if $s < 0$, and $|s|$ is the number of wavelengths that fit a circle of latitude. The form $e^{i(s\phi+\sigma t)}$ will be used in the present paper (Equ. 2.21). The suffix n is introduced as an identifying integer that is assigned according to an adopted scheme of notation; then σ, s and n specify a mode of oscillation. In the present paper σ is taken as fixed and only s and n appear as suffices, e.g. h_n^s. In order to reduce the number of suffices attaching to various symbols the convention is followed from the end of §2 onwards of denoting quantities dependent on s,n or summations of such quantities by capital letters with the suffices s and n omitted, e.g. Q stands for Q_n^s: capital letters will be used exclusively for mode-dependent quantities and small letters will be used exclusively for mode-independent quantities. One exception to this rule will be the continued use of h for h_n^s and of H for atmospheric scale height.

The extension of tidal theory to thermal excitation requires the introduction of a forcing term on the right-hand side of (1.2). Siebert (1961) presented the relevant equations and investigated heating by water vapour absorption of solar radiation. If $J_n^s(x)$ denotes the modal rate of heating per unit mass of atmosphere, (1.2) becomes

$$\frac{d^2 y_n^s}{dx^2} + \left[-\frac{1}{4} + \frac{1}{h_n^s}\left(\frac{\gamma - 1}{\gamma} H + \frac{dH}{dx} \right) \right] y_n^s = G_n^s \tag{1.3}$$

where

$$G_n^s = \frac{\gamma - 1}{\gamma^2} \; \frac{J_n^s(x)\, e^{-\frac{1}{2}x}}{g\, h_n^s} \tag{1.4}$$

and g is a constant acceleration due to gravity.

For realistic profiles of scale height H, it is necessary to solve (1.2) or (1.3) numerically. Before 1950, solutions of (1.2) were obtained by hand computation and differential analysers for a variety of scale height profiles in order to derive atmospheric response curves (Wilkes, 1949; Jacchia & Kopal, 1952). Integrations were carried out from an upper boundary height between 125 and 150 km down to the surface to provide a (complex) solution y satisfying the upper boundary radiation condition. y was then multiplied by a complex constant determined by the amplitude and phase of the gravitational tidal potential through the lower boundary condition which required a vanishing vertical velocity component at the surface.

Numerical solutions of (1.3) were obtained with the aid of an electronic computer by Butler & Small (1963) in an analysis which showed heating by ozone absorption to be the dominant generator of the solar semi-diurnal barometric oscillation. Butler & Small followed the procedure of Jacchia & Kopal (1952) and obtained two real solutions y_1, y_2 of (1.2) such that $y_1 + iy_2$ was the required linear combination satisfying the upper boundary condition. The solution of (1.3) was obtained by the method of variation of parameters in the form

$$y_n^\sigma = (\alpha + i\beta)(y_1 + i y_2) + I_n^\sigma \tag{1.5}$$

where

$$I_n^\sigma(x) = y_1(x)\int_\infty^x \left[\frac{y_2 G_n^\sigma}{y_1' y_2 - y_1 y_2'}\right]_\xi d\xi - y_2(x)\int_\infty^x \left[\frac{y_1 G_n^\sigma}{y_1' y_2 - y_1 y_2'}\right]_\xi d\xi \tag{1.6}$$

and dashes denote differentiation. $\alpha + i\beta$ was determined as previously to give a vanishing vertical component of velocity at the surface (with the tidal potential now disregarded).

The availability of high-speed computers has been a major factor in reducing to manageable proportions the task of investigating thermal atmospheric tides for different choices of H and J_n^s. Accordingly, Lindzen (1968) integrated (1.3) as it stands by a method that numerically applied the required upper and lower boundary conditions: the corresponding tidal wind and temperature fields as well as surface pressure were also computed.

The present paper is concerned with developing the analytical approach which has previously led to (1.5) and to the following equation for the surface pressure oscillation (Butler & Small, 1963)

$$p_n^s = -\frac{\gamma p_{00}}{\sigma}\left[\frac{(y_1+iy_2)(dI_n^s/dx) - I_n^s\,(d/dx)(y_1+iy_2)}{(H/h_n^s + d/dx - \frac{1}{2})(y_1+iy_2)}\right]_{x=0} \qquad (1.7)$$

On examining the terms in (1.7) it is seen by (1.4), (1.6) that the heating rate $J_n^s(x)$ contributes differentially to the surface pressure through the integral I_n^s. Weighting functions $W_p(z)$ may therefore be defined for a given mode which weight $J_n^s(x(z))$ in proportion to its differential contribution to the surface pressure oscillation (Groves, 1975). In a similar way thermal response weighting functions $W_t(z)$ have been introduced (Groves, 1975, 1976, 1977) which, for a given mode, weight $J_n^s(x(z))$ in proportional to its differential contribution to tidal fields of wind, temperature or pressure at greater heights. The derivation of w_p and w_t has not previously been given and is included in §§ 10 and 11.

The simplifying assumptions under which (1.3) has been derived are detailed by Chapman & Lindzen (1970): the more significant approximations are considered to be the neglect of (i) the Earth's topography, (ii) dissipation processes, (iii) non-linear effects and (iv) winds and temperature gradients in the unperturbed atmosphere. The analysis developed within this framework of assumptions has been termed classical tidal theory (Lindzen, 1968). The main advantage of the theory over more general treatments is the considerable mathematical simplification that arises from the separation of vertical and horizontal dependences with respect to each mode of oscillation. Tidal motion in general may then be represented by a summation of such modes which propagate independently of each other. One notable extension of classical theory which preserves separability has been the inclusion of Newtonian cooling, i.e. a rate of energy loss that is proportional to the temperature perturbation (Lindzen & McKenzie, 1967; Lindzen, 1968). This form of dissipation will be included in the present analysis.

2. Expressions for horizontally-dependent terms

Before proceeding to the treatment of vertical structure, series expansions of horizontally-dependent terms employed in classical tidal theory will be developed. The series coefficients introduced become the vertical functions involved in the later analysis.

Let Q_o be an atmospheric parameter which varies periodically with constant angular frequency σ (>0) and let

$$\mu = \cos\theta \tag{2.1}$$

where θ is colatitude. We omit the height dependence and write

$$Q_o(\mu, \phi, t) = \ell_Q\left[Q_c^R(\mu,\phi)\cos\sigma t + Q_c^I(\mu,\phi)\sin\sigma t\right] \tag{2.2}$$

where ϕ is longitude, t is time and ℓ_Q is a constant having the same physical dimensions as Q_o. We define

$$Q_c = Q_c^R + i\,Q_c^I \tag{2.3}$$

then

$$Q_o = \ell_Q \,\mathrm{Rl}\, Q_c^* e^{i\sigma t} \tag{2.4}$$

<u>Notation</u>: Superscripts R and I are used to denote the real and imaginary parts of a complex number and an asterisk its complex conjugate.

Q_c is periodic in ϕ in the interval $(0,2\pi)$ and may be expanded as a Fourier series

$$Q_c = \sum_{s=0}^{\infty}\left(Q_C^s \cos s\phi + Q_S^s \sin s\phi\right) \tag{2.5}$$

where Q_C^s, Q_S^s are complex. It is convenient to replace Q_C^s, Q_S^s

by $Q^{\pm s}$ defined by

$$Q^s = Q_C^s \qquad (s=0)$$

$$2Q^s = Q_C^s + i\,Q_S^s \qquad (s>0) \qquad (2.6)$$

$$2Q^{-s} = Q_C^s - i\,Q_S^s \qquad (s>0)$$

Then (2.5) becomes

$$Q_c = \sum_{s=-\infty}^{\infty} Q^s e^{-is\phi} \qquad (2.7)$$

and (2.4) becomes

$$Q_o = \ell_Q \, \mathrm{Rl} \sum_{s=-\infty}^{\infty} Q^{s*} e^{i(s\phi+\sigma t)} \qquad (2.8)$$

Terms in (2.8) having s positive (negative) represent a westward (eastward) progression of phase. In the special case when $Q^s = 0 \quad (s \neq s_o)$, (2.8) becomes

$$Q_o = \ell_Q \left(Q^{s_o R} \cos \sigma t' + Q^{s_o I} \sin \sigma t' \right) \qquad (2.9)$$

where

$$t' = s_o \phi/\sigma + t \qquad (2.10)$$

t' is then local mean solar time if σ/s_o is equal to the solar rate of rotation of the Earth and t is Greenwich mean solar time.

Q^s is a function of latitude and in classical tidal theory is expanded either in terms of Hough functions Θ_n^s or of Θ_{Qn}^s that are related to Θ_n^s:

$$Q^{s} = \sum_{n} Q_{n}^{s} \Theta_{n}^{s}(\mu) \qquad (Q = W, T, P, \Omega, J) \qquad (2.11)$$

$$= \sum_{n} Q_{n}^{s} \Theta_{Qn}^{s}(\mu) \qquad (Q = U, V) \qquad (2.12)$$

where, for Q = U,V,W,T or P, the quantity Q_o is the perturbation of the eastward, northward, vertically upward components of wind velocity, temperature or pressure respectively. For $Q = \Omega, J$, Q_o refers to the forcing function Ω_o, which is the potential of an applied force per unit mass of atmosphere $(= -\nabla\Omega_o)$, and to J_o, which is the rate of diabatic heating per unit mass of atmosphere. The summation in (2.11) is taken over all members of the set of Θ_n^s satisfying Laplace's tidal equation

$$\frac{d}{d\mu}\left[\frac{1-\mu^2}{f^2-\mu^2}\frac{d\Theta_n^s}{d\mu}\right] - \frac{1}{f^2-\mu^2}\left[\frac{s}{f}\frac{f^2+\mu^2}{f^2-\mu^2} + \frac{s^2}{1-\mu^2}\right]\Theta_n^s + \frac{4a_o^2\omega_o^2}{g_o h_n^s}\Theta_n^s = 0 \qquad (2.13)$$

for $-1 \leqslant \mu \leqslant 1$, where

$$f = \sigma/2\omega_o \qquad (2.14)$$

and ω_o is the Earth's sidereal rate of rotation, a_o its radius and g_o the surface acceleration due to gravity. Θ_n^s is the normalized eigenfunction of (2.13) corresponding to the eigenvalue h_n^s. Methods for calculating h_n^s and Θ_n^s have previously been reviewed (Groves, 1979).

Horizontal wind oscillations depend on horizontal gradients of $\Omega_0 + P_0/\rho_0$, where ρ_0 is unperturbed air density, and their latitudinal dependence takes the form of (2.12) where

$$\Theta_{Un}^{s} = \frac{(1-\mu^2)^{\frac{1}{2}}}{f^2-\mu^2}\left[\frac{s}{1-\mu^2} - \frac{\mu}{f}\frac{d}{d\mu}\right]\Theta_n^s \tag{2.15}$$

$$\Theta_{Vn}^{s} = \frac{(1-\mu^2)^{\frac{1}{2}}}{f^2-\mu^2}\left[\frac{(s/f)\mu}{1-\mu^2} - \frac{d}{d\mu}\right]\Theta_n^s \tag{2.16}$$

A method for obtaining Θ_{Qn}^{s} (Q = U,V) by series expansions which avoid the indeterminacy of (2.15), (2.16) when $\mu = f$ has previously been given (Groves, 1979).

Hough functions belonging to the same set are orthogonal and when normalized

$$\int_{-1}^{1} \Theta_n^s \Theta_{n'}^s \, d\mu \begin{array}{ll} = 0 & (n \neq n') \\ = 1 & (n = n') \end{array} \tag{2.17}$$

By (2.4), (2.7) and (2.17), the coefficients in (2.11) may then be obtained from Q_0 as

$$Q_n^s = \frac{\sigma}{2\pi^2 l_Q}\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2\pi/\sigma} Q_0(\mu,\phi,t)\,\Theta_n^s(\mu)\,e^{i(s\phi+\sigma t)}\,dt\,d\phi\,d\mu \tag{2.18}$$

for Q = W,T,P,Ω and J. An alternative expression that follows from (2.18) on replacing σt by $\sigma t - \frac{\pi}{2}$ is

$$i\,Q_n^s = \frac{\sigma}{2\pi^2 l_Q}\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2\pi/\sigma} Q_0'(\mu,\phi,t)\,\Theta_n^s(\mu)\,e^{i(s\phi+\sigma t)}\,dt\,d\phi\,d\mu \tag{2.19}$$

where by (2.1)

$$Q_0'(\mu,\phi,t) \equiv Q_0(\mu,\phi,t-\pi/2\sigma) = \ell_Q\left(Q_c^R \sin\sigma t - Q_c^I \cos\sigma t\right) \qquad (2.20)$$

The height dependence of Q_0 is expressed by Q_n^s.

<u>Notation</u>: Capital letters will be used solely and exclusively for quantities dependent on s and n, the suffices s and n being omitted, and for summations of such quantities. Capital letters having suffices r,j (=1,2), e.g. A_{rj}, C_{rj}, D_{oj}, E_r, I_r, K_r, P_r, Y_{oj}, Y_{aj}, Y_{oj}^{Ω} and Ξ_{oj}, are also dependent on s and n. Exceptions are made with h_n^s, scale height H and the gas constant for air R_M.

By (2.4), (2.7), (2.11) and (2.12), the expansion of Q_0 is

$$Q_0 = \ell_Q \, Rl \sum_{s,n} Q^* \Theta_{(Q)} e^{i(-s\phi+\sigma t)} \qquad (2.21)$$

where (Q) denotes Q for Q = U,V or is otherwise omitted.

The factors ℓ_Q are chosen as follows

$$\begin{aligned} \ell_T &= a_0 g_0 / 2f R_M & \ell_P &= p_{c0}/2f \\ \ell_U &= \ell_V = g_0/4\omega_0 & \ell_W &= a_0\omega_0 \\ \ell_\Omega &= a_0 g_0/2f & \ell_J &= a_0 g_0 \omega_0 \\ \ell_L &= a_0 g_0 \omega_0 \gamma / R_M & \ell_E &= \tfrac{1}{4}\ell_P \ell_W \end{aligned} \qquad (2.22)$$

ℓ_L and ℓ_E will be introduced in §§ 3 and 4 respectively.

3. Vertical structure relations

The equations of classical tidal theory are taken from Chapman & Lindzen (1970), where the height dependent functions u_n, v_n, w_n, δT_n, δp_n, Ω_n, J_n are related to the quantities U, V, W, T, P, Ω, J introduced in § 2 by (2.21) as follows

$$u_n = \ell_U U^* \qquad v_n = \ell_V V^* \qquad w_n = \ell_W W^*$$

$$\delta T_n = \ell_T T^* \qquad \delta p_n = \ell_P P^* \tag{3.1}$$

$$\Omega_n = \ell_\Omega \Omega^* \qquad J_n = \ell_J J^*$$

A dependent variable Y is introduced here to replace that denoted by y in (1.2), by y_n^s in (1.3) and by y_n in Chapman & Lindzen (1970). We write

$$y_n = (a_0 \omega_0 / \gamma' h) Y^* \tag{3.2}$$

In place of U, V, W, T, P we begin by working in terms of

$$\begin{aligned} Y^U &= -iU \\ Y^V &= V \\ Y^W &= W - i\Omega \\ Y^T &= i(1 + ia_T)T + \kappa J - i\lambda\Omega \\ Y^P &= i\left[(e^x H/a_0)P + \Omega\right] \end{aligned} \tag{3.3}$$

where

$$\lambda = dH/dz \tag{3.4}$$

$$\kappa = (\gamma - 1)/\gamma \tag{3.5}$$

a_T in (3.3) is a dimensionless quantity related to the rate coefficient of Newtonian cooling denoted by a in Chapman & Lindzen (1970). We have

$$a_T = a/\sigma\gamma' \tag{3.6}$$

Then L_o, the rate of decrease of the temperature perturbation T_o, is given by

$$L_o/\ell_L = a_T T_o/\ell_T \tag{3.7}$$

where ℓ_L is defined by (2.22). For a single mode (2.18) and (3.7) give

$$L = a_T T \tag{3.8}$$

Classical tidal theory relates Y^Q to Y by

$$Y^Q = e^{\frac{1}{2}x}\mathcal{F}_Q(Y) \quad (Q = U, V, W, T, P) \tag{3.9}$$

where

$$\mathcal{F}_U = \mathcal{F}_V = \mathcal{F}_P = \frac{1}{2} - \frac{d}{dx}$$

$$\mathcal{F}_W = (H/h) - \mathcal{F}_P \tag{3.10}$$

$$\mathcal{F}_T = (\kappa + \lambda)(H/h) - \lambda\mathcal{F}_P$$

From (3.9) and (3.10)

$$e^{\frac{1}{2}x} Y = (\hbar/H)(Y^W + Y^P) \qquad (3.11)$$

$$Y^U = Y^V = Y^P \qquad (3.12)$$

$$Y^T = \kappa Y^P + (\kappa + \lambda) Y^W \qquad (3.13)$$

Y satisfies the vertical structure equation (Chapman & Lindzen, 1970) which may be written as

$$\mathcal{D}(Y) = e^{-\frac{1}{2}x}\left[\kappa J/(1 + i a_T) + 2 i \psi \Omega\right] \qquad (3.14)$$

where

$$\mathcal{D} = d^2/dx^2 - 2\psi \, d/dx + R \qquad (3.15)$$

$$2\psi = i \lambda a_T/(1 + i a_T) \qquad (3.16)$$

$$R = F - \tfrac{1}{4} + \psi \qquad (3.17)$$

$$F = (H/\hbar)(\kappa + \lambda)/(1 + i a_T) \qquad (3.18)$$

4. Vertical energy flux

The rate of flow of wave energy in a vertical direction in a column of constant cross-section receives prominent attention in tidal theory in connection with the formulation of an upper boundary condition. As the time average of first-order flux terms is zero, products of first-order terms need to be retained. Following Wilkes (1949) the time-averaged vertically upward energy flux at a given height in the notation of the present paper is

$$\bar{E}_{(t)} = \overline{P_o W_{o(t)}} \qquad (4.1)$$

where bracketed suffices denote averaged quantities. Hence by (2.4)

$$\bar{E}_{(t)} = \tfrac{1}{2}\, \ell_P \ell_W \, \mathrm{Rl}(P_c^* W_c) \qquad (4.2)$$

To obtain the global average of this quantity we first average over all longitudes to obtain by (2.7)

$$\bar{E}_{(t,\phi)} = \tfrac{1}{2}\, \ell_P \ell_W \sum_{s=-\infty}^{\infty} \mathrm{Rl}(P^{s*} W^{s}) \qquad (4.3)$$

and then average with respect to μ from -1 to +1 to obtain by (2.11) and (2.17)

$$\bar{E}_{(t,\phi,\mu)} = \ell_E \sum_{s=-\infty}^{\infty} \sum_{n} E \qquad (4.4)$$

where

$$E = \mathrm{Rl}(P^* W) \qquad (4.5)$$

and ℓ_E is defined by (2.22). By (3.3) and (3.9) it follows from (4.5) that

$$E = (-a_0/H)\,\mathrm{Im}\left[\mathcal{F}_P(Y^*) + i\Omega^* e^{-\frac{1}{2}x}\right]\left[\mathcal{F}_W(Y) + i\Omega e^{-\frac{1}{2}x}\right]$$

$$= (a_0/k)\,\mathrm{Im}\left[Y\,(dY^*/dx - i\Omega^* e^{-\frac{1}{2}x})\right] \qquad (4.6)$$

by (3.10). For $x > x_\Omega$ such that $e^{-\frac{1}{2}x_\Omega} \ll 1$, the term in Ω^* may be neglected and (4.6) reduces to

$$E = (a_0/k)\,\mathrm{Im}\,(Y\,dY^*/dx) \qquad (4.7)$$

The accuracy of this approximation is better than 1 per cent if $e^{-\frac{1}{2}x_\Omega} = 0.01$, i.e. if height exceeds about 70 km. For a thermal source of excitation, (4.7) is exact.

If in place of (4.1) we take

$$\bar{E}_{(t)} = \overline{(P_0 + \rho_0\Omega_0)(W_0 + \dot{\Omega}_0/g_0)}_{(t)} \qquad (4.8)$$

it can be shown by (1.1), (2.4), (2.7), (2.11), (2.17) and (2.22) that after a short reduction

$$E = \mathrm{Rl}\left\{\left[P^* + (a_0/He^x)\,\Omega^*\right](W - i\Omega)\right\} \qquad (4.9)$$

on putting $H = p_0 g_0/\rho_0$. Hence by (3.3), (3.9) and (3.10) it follows that (4.7) holds without approximation. (4.8) takes account of the flux of potential energy $\rho_0\Omega_0 W_0$ by replacing P_0 by $P_0 + \rho_0\Omega_0$, where ρ_0 is unperturbed air density, and the energy flux is evaluated with respect to an equipotential surface for which the vertical velocity is $-\dot{\Omega}_0/g_0$ by replacing W_0 by $W_0 + \dot{\Omega}_0/g_0$. E is then zero for the equilibrium tide.

5. General solutions for height-dependent functions

Let $Y_o = Y_{o1}, Y_{o2}$ be any two independent solutions of the homogeneous equation

$$\mathcal{D}(Y_o) = 0 \tag{5.1}$$

and let $Y = Y'$ be a particular integral of (3.14), then the general solution of (3.14) is

$$Y = \Xi_o \wedge Y_o + Y' \tag{5.2}$$

where $\Xi_o = \Xi_{o1}, \Xi_{o2}$ are arbitrary constants and $\wedge$ is defined by

Notation: $\quad a \wedge b = a_1 b_2 - a_2 b_1$

By definition the Wronskian of Y_o is

$$w_o(x) = Y_o \wedge \frac{dY_o}{dx} \tag{5.3}$$

and it follows from (3.14) and the Abel-Liouville formula that

$$w_o(x) = w_o(x_o) \exp\left[2\int_{x_o}^{x} \psi(u)\,du\right] \tag{5.4}$$

$Y_o(x_o)$, $dY_o(x_o)/dx$ may be chosen arbitrarily as initial conditions for the integration of (5.1), and hence by (5.3) we may arrange for $w_o(x_o)$ to be unity. This condition will be introduced later (Equ. 12.5) when initial conditions for the numerical integration of (5.1) are considered. For the present we note that since ψ is mode independent, $w_o(x)$ is also mode independent by (5.4).

By the method of variation of parameters a particular integral of (3.14) may be obtained as

$$Y'(x_A, x) = \int_{x_A}^{x} \left[Y_0(u) \wedge Y_0(x) \right] S(u)\, du \qquad (5.5)$$

where

$$S(u) = s_J(u)\, J(u) + s_\Omega(u)\, \Omega(u)$$

$$s_J(u) = K e^{-\frac{1}{2}u} / \left[1 + i\, a_T(u) \right] w_c(u) \qquad (5.6)$$

$$s_\Omega(u) = 2 i \psi(u) e^{-\frac{1}{2}u} / w_0(u)$$

and x_A is arbitrary. On substituting for Y' from (5.5) into (5.2) we obtain

$$Y(x) = D_0(x) \wedge Y_0(x) \qquad (5.7)$$

and hence that

$$\frac{dY(x)}{dx} = D_0(x) \wedge \frac{dY_0(x)}{dx} \qquad (5.8)$$

where

$$D_{0j}(x) = \Xi_{0j}(x_A) + \int_{x_A}^{x} Y_{0j}(u) S(u)\, du \qquad (5.9)$$

The suffix j (= 1,2) will be used exclusively for the two independent solutions of (5.1).

On substituting for Y in (3.9) from (5.7) and noting from (3.10) that the operator $\mathcal{F}_2$ is linear in d/dx, we obtain by (5.8)

$$Y^Q(x) = \mathcal{D}_o(x) \wedge Y^Q_o(x) \tag{5.10}$$

where by definition

$$Y^Q_{oj}(x) = e^{\frac{1}{2}x} \mathcal{Z}_Q(Y_{oj}) \tag{5.11}$$

From (3.10) and (5.11)

$$\begin{aligned} e^{\frac{1}{2}x} Y_{oj} &= (\ell/H)(Y^W_{oj} + Y^P_{oj}) \\ Y^U_{oj} &= Y^V_{oj} = Y^P_{oj} \\ Y^T_{oj} &= \kappa Y^P_{oj} + (\kappa + \lambda) Y^W_{oj} \end{aligned} \tag{5.12}$$

By (5.9) we may write (5.7) as

$$Y(x) + \int_{x_A}^{x} \left[Y_o(x) \wedge Y_o(u) \right] S(u)\, du = \tilde{\Xi}_o(x_A) \wedge Y_o(x) \tag{5.13}$$

and (5.10) as

$$Y^Q(x_B) + \int_{x_A}^{x_B} \left[Y^Q_o(x_B) \wedge Y_o(u) \right] S(u)\, du = \tilde{\Xi}_o(x_A) \wedge Y^Q_o(x_B) \tag{5.14}$$

on putting $x = x_B$. In (5.13), (5.14) x_A and x_B are arbitrary values.

6. WKBJ solutions for Y and E

WKBJ solutions of (5.1) are examined in this section in preparation for the formulation of the upper boundary condition in § 7. By a change of variable from Y_{oj} to

$$Y_{aj}(x) = Y_{oj}(x)\exp\left[-\int_{\xi}^{x}\psi(u)\,du\right] \tag{6.1}$$

where ξ is arbitrary, (5.1) reduces by (3.15) to

$$\left[\frac{d^2}{dx^2} + G(x)\right] Y_a(x) = 0 \tag{6.2}$$

where

$$G = F + \frac{d\psi}{dx} - (\psi - \tfrac{1}{2})^2 \tag{6.3}$$

If G were constant, the solutions of (6.2) would be sinusoidal or exponential. In general G varies with height, but if the variation is sufficiently slow, solutions of approximately sinusoidal or exponential form may be obtained.

WKBJ solutions may be formulated for a range of values of x for which

$$\left|\tfrac{1}{4}\left(G_{xx}/G\right) - \tfrac{5}{16}\left(G_x/G\right)^2\right| / |G| \ll 1 \tag{6.4}$$

where suffix x denotes differentiation with respect to x. Solutions of (6.2) are then approximated by $\left[G(x)\right]^{-1/4}\exp\left\{\pm i\int_{\xi}^{x}\left[G(u)\right]^{1/2}du\right\}$. Two independent WKBJ solutions of (5.1) which satisfy initial conditions

$$Y_{oj}(\xi) = a_j$$
$$\frac{dY_{oj}(\xi)}{dx} = a_j' \qquad (6.5)$$

are given by

$$[K_2(\xi) - K_1(\xi)]\, Y_{oj}(x) = [i a_j' + K_2(\xi)\, a_j] \exp[i I_1(\xi, x)]$$
$$- [i a_j' + K_1(\xi) a_j] \exp[i I_2(\xi, x)] \qquad (6.6)$$

on using (6.1), where

$$I_r(\xi, x) = \int_\xi^x K_r(u)\, du \qquad (6.7)$$

$$K_1 = \Gamma_1 - i\Gamma_2 \qquad K_2 = -\Gamma_1 - i\Gamma_2 \qquad (6.8)$$

$$\Gamma_1 = G^{\frac{1}{2}} \quad (-\tfrac{\pi}{2} < \arg \Gamma \leqslant \tfrac{\pi}{2}) \qquad \Gamma_2 = \psi - G_x/4G \qquad (6.9)$$

The suffix r (= 1,2) will be used exclusively for terms associated with the two WKBJ exponential forms.

We define

$$A_{rj}(x) = \left[i \frac{dY_{oj}(x)}{dx} + K_{r'}(x)\, Y_{oj}(x) \right] \Big/ \left[K_{r'}(x) - K_r(x) \right] \qquad (6.10)$$

where

$$r' = 3 - r \qquad (6.11)$$

and Y_{oj}, dY_{oj}/dx are derived from (6.6). Then (6.6) may be written by (6.5) as

$$Y_{0j}(x) = A_{1j}(\xi)\exp\left[iI_1(\xi,x)\right] + A_{2j}(\xi)\exp\left[iI_2(\xi,x)\right] \qquad (6.12)$$

Putting $\xi = x$ in (6.12) we obtain

$$A_{rj}(x) + A_{r'j}(x) = Y_{0j}(x) \qquad (6.13)$$

From (6.10) and (6.12) with $x = \xi'$, it follows that

$$A_{rj}(\xi') = A_{rj}(\xi)\exp\left[iI_r(\xi,\xi')\right] \qquad (6.14)$$

and from (6.14) that

$$\frac{A_{r1}(\xi')}{A_{r2}(\xi')} = \frac{A_{r1}(\xi)}{A_{r2}(\xi)} \qquad (6.15)$$

We define

$$C_{r1} = \frac{A_{r1}(x)}{A_{r2}(x)} = \frac{i\dfrac{dY_{01}(x)}{dx} + K_{r'}(x)Y_{01}(x)}{i\dfrac{dY_{02}(x)}{dx} + K_{r'}(x)Y_{02}(x)}$$
$$C_{r2} = 1 \qquad (6.16)$$

then by (6.15) C_{r1} is independent of x.

From (5.7) and (6.12), we obtain

$$Y(x) = P_1(\xi,x)\exp\left[iI_1(\xi,x)\right] + P_2(\xi,x)\exp\left[iI_2(\xi,x)\right] \qquad (6.17)$$

$$\frac{dY(x)}{dx} = iK_1(x)P_1(\xi,x)\exp\left[iI_1(\xi,x)\right] + iK_2(x)P_2(\xi,x)\exp\left[iI_2(\xi,x)\right] \qquad (6.18)$$

where

$$P_r(\xi,x) = D_0(x)\wedge A_r(\xi) \qquad (6.19)$$

Putting $\xi = x$ in (6.17) we obtain

$$P_r(x,x) + P_{r'}(x,x) = Y(x) \qquad (6.20)$$

It follows from (6.14) and (6.19) that

$$P_r(\xi',x) = P_r(\xi,x)\exp\left[i\, I_r(\xi,\xi')\right] \qquad (6.21)$$

and hence from (6.17) that Y(x) is independent of ξ.

On substituting (6.17), (6.18) into (4.7), we obtain by (6.21) with $\xi' = x$

$$E = E_1 + E_2 + E' \qquad (6.22)$$

where

$$E_r(x) = -(a_0/h)K_r^R(x)\left|P_r(\xi,x)\right|^2\exp\left[-2\,I_r^I(\xi,x)\right] \qquad (6.23)$$

$$= -(a_0/h)K_r^R(x)\left|P_r(x,x)\right|^2 \qquad (6.24)$$

$$E'(x) = -(a_0/h)\,\mathrm{Rl}\left\{\left[K_1^*(x) + K_2(x)\right]P_1^*(x,x)\,P_2(x,x)\right\} \qquad (6.25)$$

E_r is an upward or downward flux according to whether $-K_r^R/h$ is positive or negative. E' arises from an interaction between the two waveforms whose sum is Y in (6.17).

7. Formulation of the upper boundary condition

Let x_L be the upper limit to the range of x for which a solution for Y is sought. To obtain a boundary condition at $x = x_L$ consideration needs to be given to the properties of the atmosphere at $x > x_L$. Following previous accounts (Wilkes, 1949; Butler & Small, 1963; Lindzen, 1968) we assume that (1) the energy flux remains bounded as $x \to \infty$ and (2) the radiation condition holds, which means that there is no incoming energy at large values of x. The radiation condition is usually applied on the assumption that H and a_T are constant, but the following assumption is less restrictive. We assume that a value of x_L ($\geqslant x_S$) can be found such that:

(i) $S = 0 \quad (x \geqslant x_S)$ (7.1)

(ii) WKBJ solutions are valid for $x \geqslant x_L$

(iii) For $x \geqslant x_L$, either

(a) $|\Gamma_1^I| > |\Gamma_2^R|$ (7.2)

or (b) $|\Gamma_1^R| > |\Gamma_2^I|$ (7.3)

By (5.6), (i) requires that for $x \geqslant x_S$

$$s_J J = 0 \qquad (7.4)$$

$$s_\Omega \Omega = 0 \qquad (7.5)$$

as the two sources of excitation are independent. (7.4) is readily satisfied by taking x_S at a height above the region of heating whose effect is being investigated. For example the combined effects of tropospheric, stratospheric and mesospheric heating could be investigated by taking x_S at 90 km altitude; or the contribution of tropospheric heating alone could be investigated by taking x_S at say 15 km altitude. (7.5) requires, by (5.4) and (5.6), that either $\psi = 0$ or $s_\Omega \doteqdot 0$ to an acceptable order of accuracy for $x \geqslant x_S$.

Applying (i) to (5.9) gives

$$D_{o_j}(x) = D_{o_j}(x_S) \qquad (x \geqslant x_S) \qquad (7.6)$$

Then (6.19) with $\xi = x_L$ and (7.6) give

$$P_r(x_L, x) = P_r(x_L, x_S) \qquad (x \geqslant x_L) \qquad (7.7)$$

Hence under (i) and (ii), (6.17) holds and becomes by (7.7) on putting $\xi = x_L$

$$Y(x) = P_1(x_L, x_S)\exp\left[i I_1(x_L, x)\right] + P_2(x_L, x_S)\exp\left[i I_2(x_L, x)\right] \qquad (x \geqslant x_L) \qquad (7.8)$$

From (6.8) we have

$$\begin{aligned} K_1^R &= \Gamma_1^R + \Gamma_2^I & \qquad K_1^I &= \Gamma_1^I - \Gamma_2^R \\ K_2^R &= -\Gamma_1^R + \Gamma_2^I & \qquad K_2^I &= -\Gamma_1^I - \Gamma_2^R \end{aligned} \qquad (7.9)$$

Hence under (iii) (a)

$$K_1^I K_2^I < 0 \qquad (x \geqslant x_L) \qquad (7.10)$$

and we choose r_o (= 1 or 2) such that $K_{r_o}^I < 0$ and $K_{r_o'}^I > 0$ where r_o' is given by (6.11). By (6.7), (6.23) and (7.7) E_{r_o} is unbounded as $x \to \infty$ unless

$$P_{r_0}(x_L, x_S) = 0 \qquad (7.11)$$

(7.11) provides an upper boundary condition for the solution of Y in the region $x_0 \leqslant x \leqslant x_L$.

Under condition (iii) (b), we have

$$K_1^I K_2^I \geqslant 0 \qquad (x \geqslant x_L) \qquad (7.12)$$

as (iii) (a) does not now hold; and by (7.9) that

$$(K_1^R/h)(K_2^R/h) < 0 \qquad (x \geqslant x_L) \quad (7.13)$$

The requirement for a bounded energy flux excludes both K_1^I and K_2^I from being negative and (7.12) therefore yields $K_1^I \geqslant 0$ and $K_2^I \geqslant 0$. By (7.13) the terms in (7.8) are then associated with either upward or downward finite energy fluxes, which by (6.23) with $\xi = x_L$ and (7.7) are

$$E_r = -(a_0/h)\, K_r^R(x)\, |P_r(x_L, x_S)|^2 \exp\left[-2 I_r^I(x_L, x)\right] \qquad (7.14)$$

We choose r_0 (= 1 or 2) such that $K_{r_0}^R/h > 0$ and $K_{r_0'}^R/h < 0$, then by (7.14) E_{r_0} is a downward flux and $E_{r_0'}$ an upward flux. Under the assumption of the radiation condition, we require $E_{r_0} = 0$ for large x. Hence by (7.14) $P_{r_0}(x_L, x_S) = 0$ and the upper boundary condition is again given by (7.11).

By (5.9) and (6.19), we can write (7.11) as

$$\int_{x_A}^{x_S} \left[A_{r_0}(x_L) \wedge Y_0(u)\right] S(u)\, du = \Xi_0(x_A) \wedge A_{r_0}(x_L) \qquad (7.15)$$

On dividing by $A_{r_0 2}(x_L)$, (7.15) becomes by (6.16)

$$\int_{x_A}^{x_S} \left[C_{r_0} \wedge Y_0(u) \right] S(u)\, du = \Xi_0(x_A) \wedge C_{r_0} \tag{7.16}$$

An upper boundary condition is provided by (7.16) with the appropriate choice of r_0, i.e. $r_0 = 1$ or 2, such that either $K^I_{r_0}(x) < 0$ (under (iii) (a)) or $K^R_{r_0}(x)/h > 0$ (under (iii) (b)) for $x \geqslant x_L$.

At this stage we are able to identify (1.5) with the general solution (5.13) subject to the condition (7.16). By (5.9)

$$D_{0j}(x_L) = \Xi_{0j}(x_A) + \int_{x_A}^{x_L} Y_{0j}(u)\, S(u)\, du = \Xi_{0j}(x_L) \tag{7.17}$$

Hence, since $S(u) = 0$ for $x \gg x_S$, (7.16) becomes

$$\Xi_0(x_L) \wedge C_{r_0} = 0 \tag{7.18}$$

and, on replacing x_A by x_L, (5.13) becomes

$$Y(x) = \Xi_0(x_L) \wedge Y_0(x) + \int_x^{x_S} \left[Y_0(x) \wedge Y_0(u) \right] S(u)\, du \tag{7.19}$$

Then, since $C_{r_0 2} = 1$, (7.18) and (7.19) give

$$Y(x) = \Xi_{02}\left[C_{r_0 1} Y_{02}(x) - Y_{01}(x) \right] + \int_x^{x_S} \left[Y_0(x) \wedge Y_0(u) \right] S(u)\, du \tag{7.20}$$

By (3.2) we may identify the complex conjugate of (7.20) with (1.5) by writing

$$\begin{aligned} a_T &= \psi = 0 & \qquad \Xi^*_{02} &= -(\alpha + i\beta) \\ Y^*_{01} &= (\gamma k / a_0 \omega_c)\, y_1 & \qquad Y^*_{02} &= (\gamma k / a_0 \omega_c)\, y_2 \end{aligned} \tag{7.21}$$

$$S^* = (a_0 \omega_0 / \gamma h)\, \mathcal{G}_n^{\delta} / (y_1 y_2' - y_2 y_1') \tag{7.22}$$

$$C^*_{r_0 1} = -i \tag{7.23}$$

(7.22) is in accord with (1.4), (2.22), (3.1), (3.2) and (5.3), but (6.16) imposes a condition on $y_1(x)$, $y_2(x)$ for $x \gg x_L$, which by (7.23) becomes

$$\frac{d}{dx}(y_1 + i y_2) = -i K^*_{r_0'} (y_1 + i y_2) \tag{7.24}$$

Hence

$$y_1 + i y_2 \propto \exp\left[-i I^*_{r_0'}\right] \qquad (x \gg x_L) \tag{7.25}$$

For an atmospheric 'top' (i.e. $x \gg x_L$) having constant scale height H and no dissipation, i.e. $a_T = \psi = 0$, we have by (3.4), (3.18) and (6.3) that G is the real constant $K(H/h) - \frac{1}{4}$. If $G < 0$, (7.9) and condition (iii) (a) require $r_0 = 2$ and (7.25) gives

$$y_1 + i y_2 \propto \exp\left[-(-\mathcal{G})^{\frac{1}{2}} x\right] \tag{7.26}$$

If $G > 0$, (7.9) and condition (iii) (b) require $r_0 = 1$ and (7.25) gives

$$y_1 + i y_2 \propto \exp\left(-i\, \mathcal{G}^{\frac{1}{2}} x\right) \tag{7.27}$$

(7.26) and (7.27) are the forms of solution introduced at high level by Wilkes (1949) for this type of atmospheric 'top'. In general (7.24) with $x = x_L$ provides the upper boundary condition in a form that may be applied to a numerical integration scheme (Lindzen, 1968).

It is now possible to express the upward energy flux at $x \geqslant x_L$ in terms of the WKBJ solution for Y. When (7.11) holds, (7.7) gives

$$P_{r_o}(x_L, x) = 0 \qquad (x \geqslant x_L) \qquad (7.28)$$

and hence from (6.21) with $\xi = x_L$, $\xi' = x$

$$P_{r_o}(x, x) = 0 \qquad (x \geqslant x_L) \qquad (7.29)$$

Therefore by (6.20)

$$P_{r_o'}(x, x) = Y(x) \qquad (x \geqslant x_L) \qquad (7.30)$$

and the upward energy flux (in units of ℓ_E) may be written by (6.24) as

$$E_{r_o'}(x) = -(a_o/h)\,K^R_{r_o'}(x)\,|Y(x)|^2 \qquad (x \geqslant x_L) \qquad (7.31)$$

where $K^R_{r_o'}/h < 0$.

8. The Y^Q relations

Arbitrary constants of integration Ξ_{01}, Ξ_{02} which were introduced in § 5 have since been retained in the analysis and in particular in the upper boundary condition (7.16). We now turn to their elimination by introducing another relation such as a lower boundary condition. We take the general form of boundary condition expressed by (5.14) and defer its identification with the lower boundary until the last paragraph of this section.

From (5.13), (5.14) and (7.16) on eliminating $\Xi_{01}(x_A)$, $\Xi_{02}(x_A)$ we obtain

$$\begin{vmatrix} Y(x)+\int_{x_A}^{x}\left[Y_o(x)\wedge Y_o(u)\right]S(u)\,du & Y_{01}(x) & Y_{02}(x) \\ Y^Q(x_B)+\int_{x_A}^{x_B}\left[Y_o^Q(x_B)\wedge Y_o(u)\right]S(u)\,du & Y_{01}^Q(x_B) & Y_{02}^Q(x_B) \\ \int_{x_A}^{x_S}\left[C_{r_0}\wedge Y_o(u)\right]S(u)\,du & C_{r_0 1} & C_{r_0 2} \end{vmatrix} = 0 \tag{8.1}$$

where Q = U, V, W, T or P, r_o = 1 or 2, and x_A, x_B are arbitrary. On putting $x_B = x_A$, (8.1) expands as

$$Y(x)+\int_{x_A}^{x}\left[Y_o(x)\wedge Y_o(u)\right]S(u)\,du = N_Q(x_A,x)\,Y^Q(x_A) + M_Q(x_A,x)\int_{x_S}^{x_A}N_Q(x_A,u)\,S(u)\,du \tag{8.2}$$

where

$$M_Q(x', x) = Y_o^Q(x') \wedge Y_o(x) \tag{8.3}$$

$$N_Q(x', x) = \left[C_{r_o} \wedge Y_o(x)\right] / \left[C_{r_o} \wedge Y_c^Q(x')\right] \tag{8.4}$$

An alternative form of (8.2) may be obtained by putting $x_A = x_S$, $x_B = x_A$ in (8.1) and expanding as

$$Y(x) + \int_{x_S}^{x} \left[Y_o(x) \wedge Y_o(u)\right] S(u)\, du = N_Q(x_A, x) \Big\{ Y^Q(x_A) + \int_{x_S}^{x_A} M_Q(x_A, u) S(u)\, du \Big\} \tag{8.5}$$

(8.5) also follows from (8.2) on using the identity

$$M_Q(x_A, u) N_Q(x_A, x) - M_Q(x_A, x) N_Q(x_A, u) \equiv Y_o(x) \wedge Y_o(u) \tag{8.6}$$

As x_A is arbitrary we may put $x_A = x$ in (8.5) and obtain

$$Y(x) = N_Q(x, x) Y^Q(x) + M_Q(x, x) \int_{x_S}^{x} N_Q(x, u) S(u)\, du \tag{8.7}$$

The introduction of an upper boundary condition therefore enables Y to be related to a single Y^α whereas (3.11) to (3.13) related Y to two different Y^α.

As the left-hand side of (8.5) is independent of x_A and Q, both of which may be assigned arbitrarily, the right-hand side may be written in two ways to give

$$N_Q(x,x)\left[Y^Q(x)+\int_{x_S}^{x} M_Q(x,u)\,S(u)\,du\right] = N_{Q'}(x_A,x)\left[Y^{Q'}(x_A) + \int_{x_S}^{x_A} M_{Q'}(x_A,u)\,S(u)\,du\right] \tag{8.8}$$

By means of the identity

$$N_{Q'}(x_A,x)M_{Q'}(x_A,u) \doteq N_Q(x,x)\left\{M_Q(x,u) + \left[Y_o^{Q'}(x_A)\wedge Y_o^{Q}(x)\right]N_{Q'}(x_A,u)\right\} \tag{8.9}$$

(8.8) may be expressed as

$$N_Q(x,x)\left\{Y^Q(x)+\int_{x_A}^{x} M_Q(x,u)\,S(u)\,du + \left[Y_o^{Q'}(x_A)\wedge Y_o^{Q}(x)\right]\int_{x_A}^{x_S} N_{Q'}(x,u)\,S(u)\,du\right\} = N_{Q'}(x_A,x)\,Y^{Q'}(x_A) \tag{8.10}$$

From (8.8) or (8.10), $Y^Q(x)$ may be evaluated for a given source function S(u) provided $Y^{Q'}(x_A)$ is known for some particular Q' = U, V, W, T or I. We choose Q' = W and introduce as a lower boundary condition at $x = x_A$ a known vertical velocity of the atmosphere expressed non-dimensionally by $W(x_A)$.

9. Formulae for the vertical dependence of oscillations

As a consequence of the linearization employed in classical tidal theory the results of § 8 show that oscillations of an atmospheric variable are independently related to the potential field, Ω, the diabatic heating, J, and the vertical motion, $W(x_A)$, at the lower boundary. For a rigid, horizontal lower boundary $W(x_A) = 0$, but more generally Earth surface tides contribute to $W(x_A)$ and the constraint imposed on air motions by an undulating terrain introduces new modes that have non-zero values for $W(x_A)$. We therefore express Q = U, V, W, T and P as the sum of the three above-mentioned contributions by

$$Q = Q_\Omega + Q_J + Q_W \qquad (9.1)$$

By (3.3), (5.6) and (8.8) with $Q' = W$ we obtain

$$U_W(x) = i\, L_U(x_A, x)\, W(x_A) \qquad (9.2)$$

$$U_\Omega(x) = L_U(x_A, x)\left[\Omega(x_A) - i\int_{x_A}^{x_S} M_{W\Omega}(x_A, u)\,\Omega(u)\,du\right] + i\int_{x}^{x_S} M_{U\Omega}(x, u)\,\Omega(u)\,du \qquad (9.3)$$

$$U_J(x) = -i\, L_U(x_A, x)\int_{x_A}^{x_S} M_{WJ}(x_A, u)\, J(u)\,du + i\int_{x}^{x_S} M_{UJ}(x_A, u)\, J(u)\,du \qquad (9.4)$$

$$V_W = -i\, U_W \qquad V_\Omega = -i\, U_\Omega \qquad V_J = -i\, U_J \qquad (9.5)$$

$$W_W(x) = L_W(x_A, x)\, W(x_A) \tag{9.6}$$

$$W_\Omega(x) = -i\, L_W(x_A, x)\left[\Omega(x_A) - i\int_{x_A}^{x_S} M_{W\Omega}(x_A, u)\,\Omega(u)\,du\right] + i\left[\Omega(x) - i\int_{x}^{x_S} M_{W\Omega}(x, u)\,\Omega(u)\,du\right] \tag{9.7}$$

$$W_J(x) = -L_W(x_A, x)\int_{x_A}^{x_S} M_{WJ}(x_A, u)\,J(u)\,du + \int_{x}^{x_S} M_{WJ}(x, u)\,J(u)\,du \tag{9.8}$$

$$\left[1 + i\,a_T(x)\right] T_W(x) = -i\, L_T(x_A, x)\, W(x_A) \tag{9.9}$$

$$\left[1 + i\,a_T(x)\right] T_\Omega(x) = -L_T(x_A, x)\left[\Omega(x_A) - i\int_{x_A}^{x_S} M_{W\Omega}(x_A, u)\,\Omega(u)\,du\right] + \lambda(x)\,\Omega(x) - i\int_{x}^{x_S} M_{T\Omega}(x, u)\,\Omega(u)\,du \tag{9.10}$$

$$\left[1 + i\,a_T(x)\right] T_J(x) = i\, L_T(x_A, x)\int_{x_A}^{x_S} M_{WJ}(x_A, u)\,J(u)\,du + i\left[\kappa\, J(x) - \int_{x}^{x_S} M_{TJ}(x, u)\,J(u)\,du\right] \tag{9.11}$$

$$\left[e^{x} H(x)/a_0\right] P_W(x) = -i\, L_P(x_A, x)\, W(x_A) \tag{9.12}$$

$$\begin{aligned}\left[e^{x} H(x)/a_0\right] P_\Omega(x) = -L_P(x_A, x) \Big[\Omega(x_A) & \\ - i\int_{x_A}^{x_S} M_{W\Omega}(x_A, u)\, \Omega(u)\, du\Big] & - \Omega(x) \\ - i\int_{x}^{x_S} M_{P\Omega}(x, u)\, \Omega(u)\, du & \end{aligned} \tag{9.13}$$

$$\begin{aligned}\left[e^{x} H(x)/a_0\right] P_J(x) = i\, L_P(x_A, x) \int_{x_A}^{x_S} M_{WJ}(x_A, u)\, J(u)\, du & \\ - i\int_{x}^{x_S} M_{PJ}(x, u)\, J(u)\, du & \end{aligned} \tag{9.14}$$

where

$$M_{QQ'}(x, u) = M_Q(x, u)\, s_{Q'}(u) \tag{9.15}$$

$$\begin{aligned} L_Q(x', x) &= N_W(x', x)/N_Q(x, x) \\ &= \left[C_{r_0} \wedge Y_c^Q(x)\right] / \left[C_{r_0} \wedge Y_c^W(x')\right] \end{aligned} \tag{9.16}$$

for Q = U, V, W, T or P and Q' = J or Ω. An alternative set of relations to (9.2) to (9.14) may be derived from (8.10) in a similar manner.

10. Oscillations at the lower boundary: surface oscillation weighting function W_p

Expressions for oscillations at the lower boundary may be obtained from (9.2) to (9.14) on putting $x = x_A$. Alternatively, such expressions may be obtained from (8.10) with $x = x_A$. The latter procedure is followed as the resulting equations simplify more readily: use is made of the relations

$$Y_o^W(x) \wedge Y_o^U(x) = \kappa^{-1}\left[Y_o^W(x) \wedge Y_o^T(x)\right] = Y_o^W(x) \wedge Y_o^P(x)$$
$$= -e^{x} H(x) \varpi_o(x)/\hbar \qquad (10.1)$$

which follow from (3.10), (5.3) and (5.11); and of

$$1 + L_P(x,x) = e^{\frac{1}{2}x} H(x) N_W(x,x)/\hbar \qquad (10.2)$$

$$\lambda(x) - L_T(x,x) = -e^{\frac{1}{2}x} \kappa H(x) N_W(x,x)/\hbar \qquad (10.3)$$

which follow from (5.12). From (8.10) we obtain by (3.3) and (9.16)

$$U_W(x_A) = i L_U(x_A, x_A) W(x_A) \qquad (10.4)$$

$$U_\Omega(x_A) = L_U(x_A, x_A)\Omega(x_A) + i\left[e^{x_A} H(x_A)\varpi_o(x_A)/\hbar\right] \times \int_{x_A}^{x_S} N_\Omega(x_A, u)\Omega(u)\, du \qquad (10.5)$$

$$U_J(x_A) = i\left[e^{x_A} H(x_A)\varpi_o(x_A)/\hbar\right]\int_{x_A}^{x_S} N_J(x_A, u) J(u)\, du \qquad (10.6)$$

$$W_W(x_A) = W(x_A) \qquad (10.7)$$

$$W_\Omega(x_A) = W_J(x_A) = 0 \qquad (10.8)$$

$$T_W(x_A) = -i\,L_T(x_A, x_A)\,W(x_A)/[1 + i\,a_T(x_A)] \tag{10.9}$$

$$\begin{aligned} T_\Omega(x_A) = -\kappa\left[e^{\frac{1}{2}x_A}H(x_A)/\hbar\right]\Big\{ & N_W(x_A, x_A)\,\Omega(x_A) \\ & + i\,e^{\frac{1}{2}x_A}\,w_0(x_A)\int_{x_A}^{x_S} N_\Omega(x_A, u)\,\Omega(u)\,du\Big\} \\ & \div\left[1 + i\,a_T(x_A)\right] \end{aligned} \tag{10.10}$$

$$\begin{aligned} T_J(x_A) = i\kappa\Big\{ J(x_A) - & \left[e^{x_A}\,w_0(x_A)\,H(x_A)/\hbar\right] \\ & \times\int_{x_A}^{x_S} N_J(x_A, u)\,J(u)\,du\Big\} \div\left[1 + i\,a_T(x_A)\right] \end{aligned} \tag{10.11}$$

$$P_W(x_A) = \left[a_0\,e^{-x_A}/i\,H(x_A)\right]L_P(x_A, x_A)\,W(x_A) \tag{10.12}$$

$$\begin{aligned} P_\Omega(x_A) = -(a_0/\hbar)\Big[& e^{-\frac{1}{2}x_A}N_W(x_A, x_A)\,\Omega(x_A) \\ & + i\,w_0(x_A)\int_{x_A}^{x_S} N_\Omega(x_A, u)\,\Omega(u)\,du\Big] \end{aligned} \tag{10.13}$$

$$P_J(x_A) = \left[a_0\,w_0(x_A)/i\hbar\right]\int_{x_A}^{x_S} N_J(x_A, u)\,J(u)\,du \tag{10.14}$$

where

$$N_Q(x, u) = N_W(x, u)\,s_Q(u) \qquad (Q = J, \Omega) \tag{10.15}$$

Equations (10.6), (10.11) and (10.14) show that $N_J(x_A, u)$ may be interpreted as a function of u that weights J(u) in the interval (u, u+du) according to its differential contribution for this interval to oscillations in horizontal velocity, temperature or pressure at the lower boundary. By (1.1) it follows that

$$W_p(z) = N_J(x_A, x(z))/H(z) \qquad (10.16)$$

weights J(x(z)) in the interval (z, z+dz) according to its differential contribution for this height interval to the lower boundary oscillations.

$W_p(z)$ has been evaluated for the migrating (s = 1) modes of solar diurnal frequency designated by n = ± 1, ± 2, . . ± 6, the corresponding values of h being taken from Chapman & Lindzen (1970). The atmosphere is assumed to be non-dissipative having $a_T = \psi = 0$ and hence by (3.18) and (6.3)

$$G = (\kappa + dH/dz)H/h - \tfrac{1}{4} \qquad (10.17)$$

κ is taken to be 2/7 corresponding to $\gamma = 7/5$. The adopted profile of scale height H is shown in Fig. 1. For negative n it is found that G is real and negative, and hence from (6.8) that K_1, K_2 are imaginary. From (6.16)

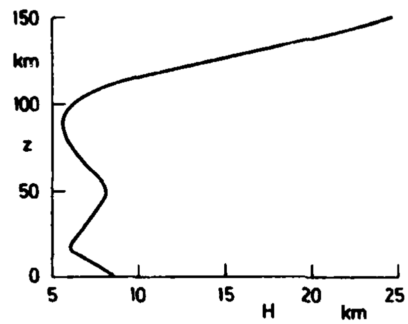

Fig. 1 Profile of scale height H.

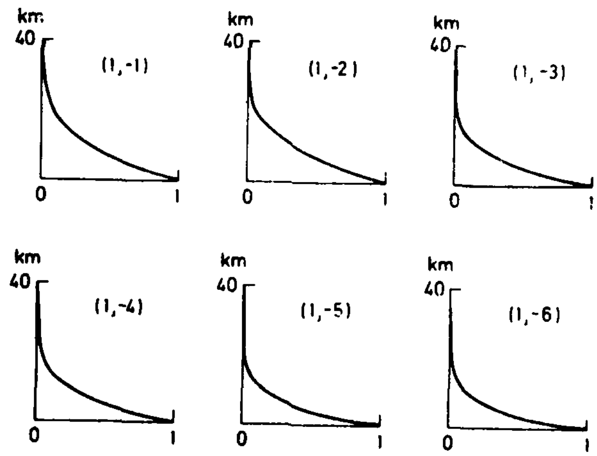

Fig. 2 The negative real part $-w_p^R$ of w_p (Equ. 10.16) plotted on an arbitrary scale for solar diurnal modes with s = 1, n = -1, . . . -6. $w_p^I = 0$. (s,n) is shown on each graph.

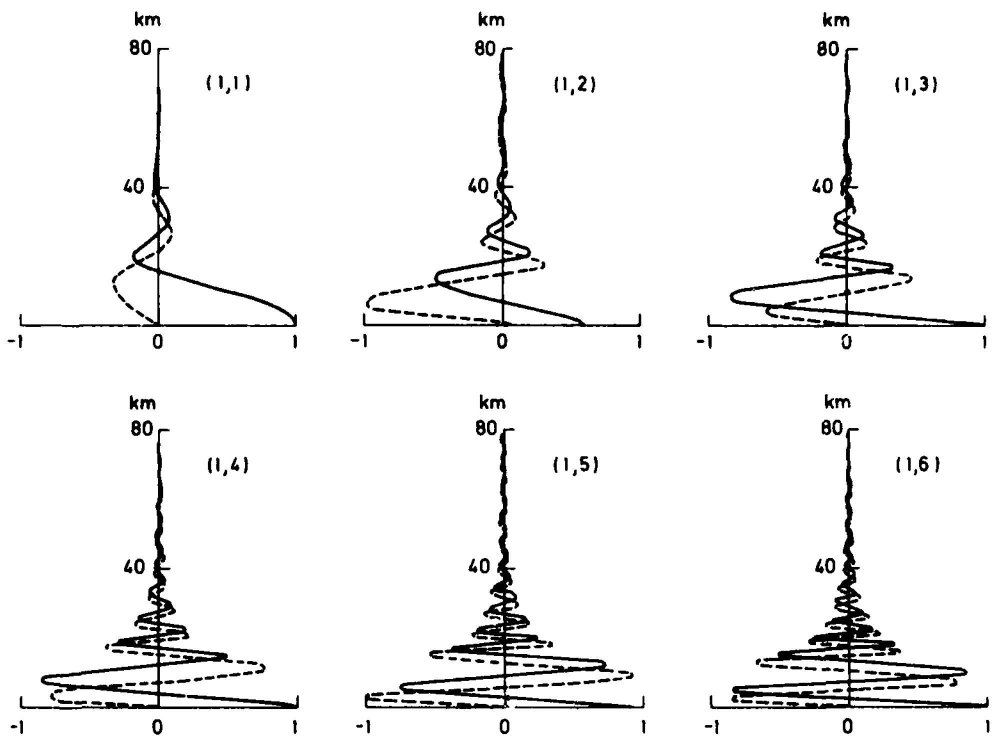

Fig. 3 Real and imaginary parts of W_p plotted on an arbitrary scale for solar diurnal modes with $s = 1$, $n = 1, \ldots 6$. Key: —— W_p^R; ----- W_p^I. (s,n) is shown on each graph.

C_{rj} are then real if $Y_{oj}(x)$ are taken to be real when solving (5.1); and hence $W_p(z)$ is real by (8.4) and (10.16). If the heating maximizes at the same time at all heights, i.e. if arg $J(x)$ is constant, it follows from (10.6), (10.11) and (10.14) that the phases of the lower boundary oscillations are in quadrature with it, i.e. they maximize earlier or later by 6 h. Fig. 2 shows $W_p(z)$ on an arbitrary scale for the first six negative migrating diurnal modes ($s = 1$, $n = -1, \ldots, -6$). Such modes are referred to as trapped modes as the generation of an oscillation by a region of heating decays in either vertical direction away from the region. The greatest contribution to an oscillation at the surface therefore arises from atmospheric heating closest to the surface, and there is an exponential-like reduction in the contribution to the surface oscillation with the height of the heating.

For positive ($n > 0$) diurnal modes $W_p(z)$ is complex and an oscillation at the surface can be resolved into components that are respectively in phase and in quadrature with the heating. The vertical structure of W_p is oscillatory dividing the atmosphere into positively and negatively weighted regions (Fig. 3). From the graph of W_p for $n = 1$ it is seen that tropospheric heating would

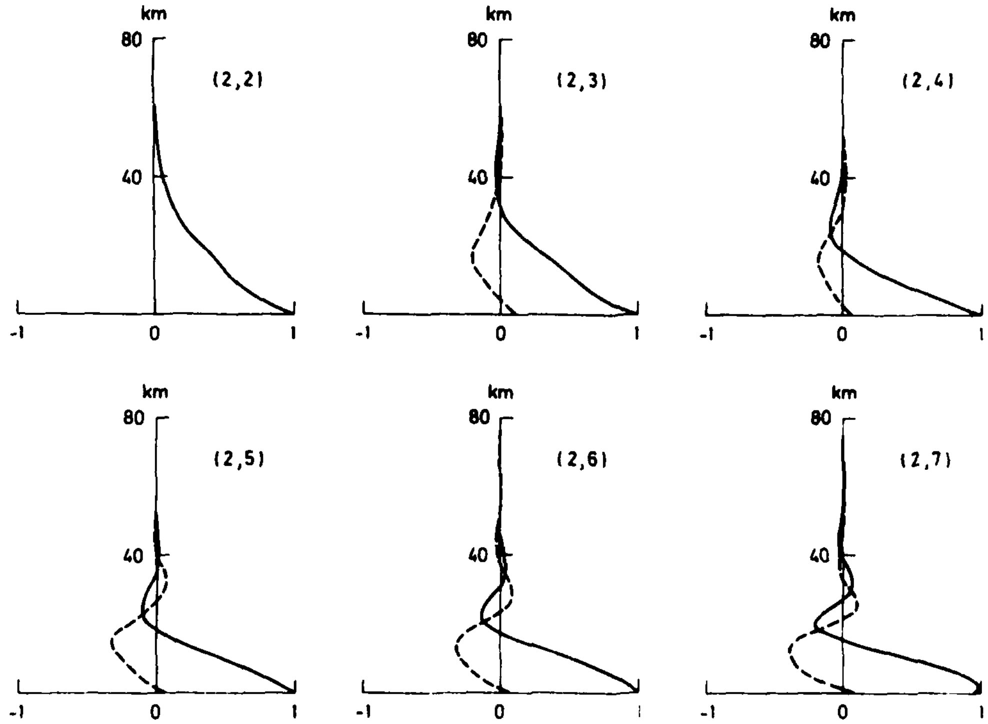

Fig. 4 Real and imaginary parts of w_p plotted on an arbitrary scale for solar semi-diurnal modes with s = 2, n = 2, . . . 7. Key: —— w_p^R; ----- w_p^I. (s,n) is shown on each graph. The imaginary part of (2,2) is negligibly small compared with the real part.

be weighted with the same sign at all heights, whereas ozone heating which extends from about 20 to 80 km is weighted by two positive and two negative regions thereby reducing its effectiveness as a generator of surface oscillations. For larger values of n, vertical wavelengths shorten and the effectiveness of tropospheric heating also becomes reduced by cancellations between positive and negative regions.

Fig. 4 shows W_p for the first six migrating ($s = 2$) semi-diurnal modes. For the leading mode ($n = 2$) W_p has the same sign at all heights and hence tropospheric heating and stratospheric heating combine in generating surface oscillations: although W_p decreases with height, ozone heating may be shown to be the main contributor to surface oscillations by adopting a typical profile of tropospheric and stratospheric heating (Groves, 1975). For increasing n, values of h decrease and W_p become increasingly oscillatory. For $n = 7$, W_p is almost identical with that for $n = 1$ in Fig. 3 as both modes have nearly the same value of h (i.e. 0.706 and 0.691 km).

11. Oscillations above a region of excitation: thermal response weighting function W_t

Above a region of excitation, expressions for the oscillations of atmospheric variables simplify. With $x \geqslant x_S$, (8.5) and (8.8) reduce to

$$Y(x) = N_W(x_A, x)\ \Phi(x_A) \qquad (11.1)$$

$$Y^Q(x) = L_Q(x_A, x)\ \Phi(x_A) \qquad (Q = U, V, W, T, P) \qquad (11.2)$$

where by (3.3), (5.6) and (9.15)

$$\Phi(x_A) = W(x_A) - i\,\Omega(x_A) - \int_{x_A}^{x_S} \left[M_{W\Omega}(x_A, u)\Omega(u) + M_{WJ}(x_A, u)J(u)\right] du \qquad (11.3)$$

Then, by (3.6) and (11.2),

$$U(x) = i\,V(x) = i\,L_U(x_A, x)\ \Phi(x_A) \qquad (11.4)$$

$$W(x) = L_W(x_A, x)\ \Phi(x_A) \qquad (11.5)$$

$$[1 + i\,a_T(x)]\,T(x) = -i\,L_T(x_A, x)\ \Phi(x_A) \qquad (11.6)$$

$$[e^x H(x)/\sigma_0]\,P(x) = -i\,L_P(x_A, x)\ \Phi(x_A) \qquad (11.7)$$

We define

$$W_t(z) = M_{WJ}(x_A, x(z))/H(z) \qquad (11.8)$$

then it follows from (11.3) that $W_t(z)$ weights $J(x(z))$ in the interval (z, z+dz) according to its differential contribution for this height interval to $\Phi(x_A)$ and hence by (11.4) to

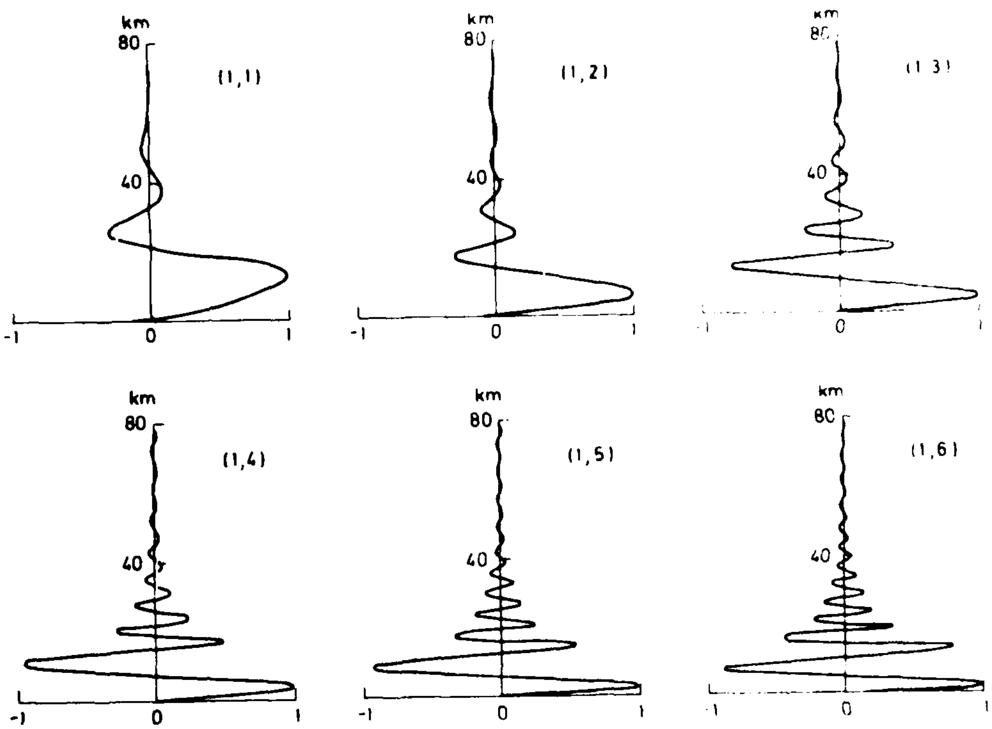

Fig. 5 The real part W_t^R of W_t (Equ. 11.8) plotted on an arbitrary scale for solar diurnal modes with $s = 1$, $n = 1, \ldots 6$. $W_t^I = 0$. (s,n) is shown on each graph.

(11.7) to the oscillations at any given $x \gg x_S$. Plots of W_t on a relative scale are shown in Figs. 5 and 6 for positive and negative solar diurnal modes with the same atmospheric data as in § 10. W_t is real for a non-dissipative atmosphere.

For positive ($n > 0$) diurnal modes, W_t is oscillatory dividing the atmosphere into height intervals which make alternate positive and negative contributions to the oscillations at a given height above the region of excitation. The greatest weights are given to excitations at the lowest heights on account of the exponential-like growth of amplitudes on propagation into air of decreasing density. Excitation by the region of ozone heating is much reduced by the cancellation of positive and negative contributions, the reduction being greater for higher values of n.

For negative n (Fig. 6), modes are characteristically trapped and above a region of excitation the greatest relative contribution to oscillations at a given height arises from the uppermost levels of the excitation: hence W_t increases with height. For $n = -1$ the trapping character is weak and the effect of a region of heating extends over a considerable range of heights.

Fig. 7 shows W_t for the first six migrating ($s = 2$) semi-diurnal modes. For $n = 2$, W_t changes sign at 15 km and therefore tropospheric heating and stratospheric heating

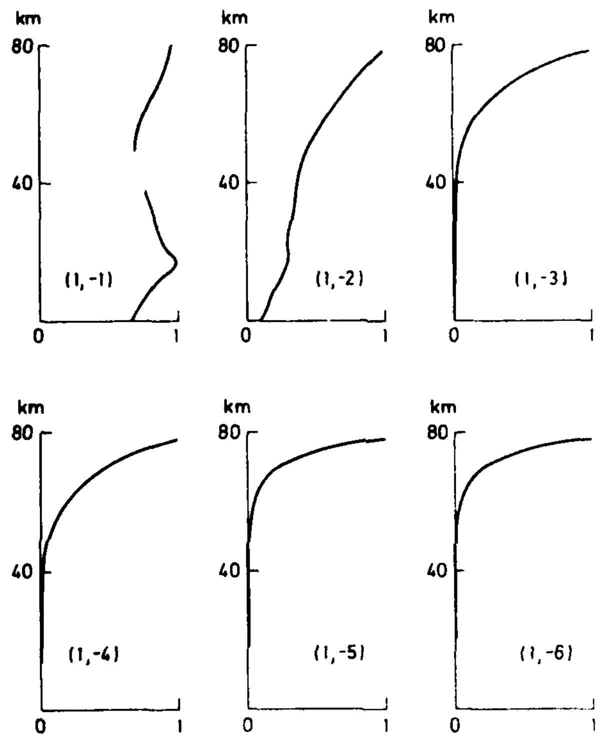

Fig. 6 As for Fig. 5 with s = 1, n = -1, . . -6.

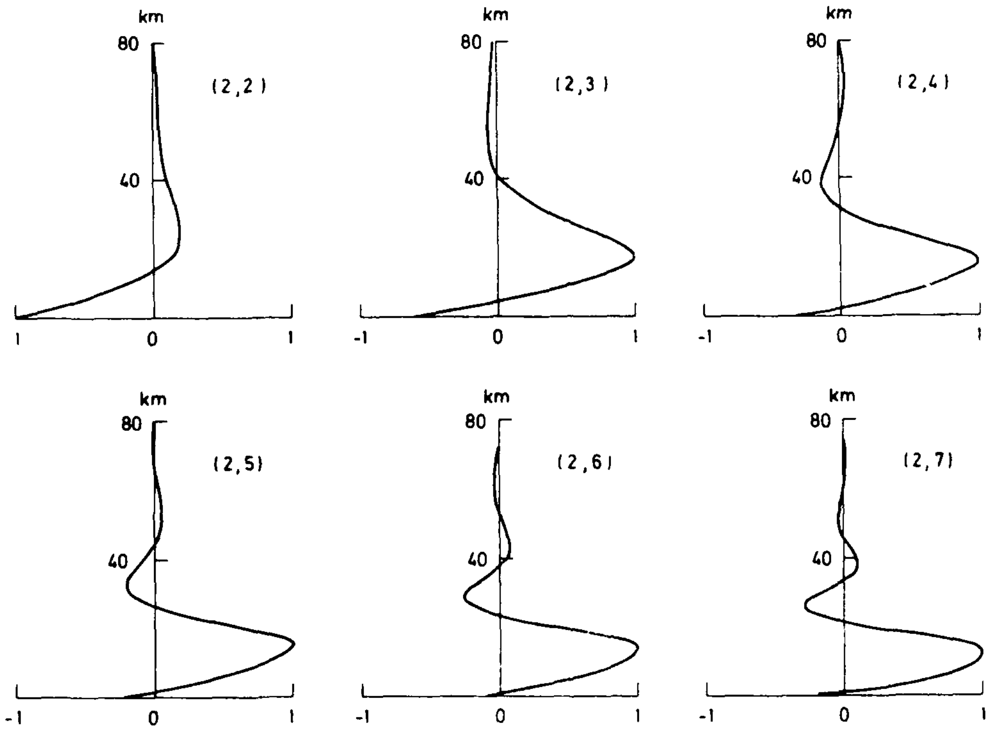

Fig. 7 The real part W_t^R of W_t plotted on an arbitrary scale for solar semi-diurnal modes with s = 2, n = 2, . . 7. $W_t^I = 0$. (s,n) is shown on each graph.

generate opposing contributions to oscillations above 80 km. For increasing n, W_t become increasingly oscillatory and the extent to which the contributions from tropospheric and stratospheric heating to oscillations above 80 km combine or oppose one another is dependent on the structure of the heating profile. For n = 7, W_t is almost the same as that for n = 1 in Fig. 5.

The upward energy flux (in units of ℓ_E) at $x \gg x_L$ has been expressed by (7.31). By (11.1) this becomes

$$E_{r_o'}(x) = -(a_0/\ell)\,K^R_{r_o'}(x)\,\left|N_W(x_A, x)\right|^2 \left|\Phi(x_A)\right|^2 \quad (x \gg x_L) \quad (11.9)$$

where $K^R_{r_o'}/h < 0$.

12. Notes on computational procedure

Let $(0, x_E)$ be the range of x for which computations are undertaken, then in order to evaluate C_{r_o} x_E needs to exceed x_L. Initially x_L is not known but values of x_E corresponding to 150 km have been found to be adequate. Two independent solutions Y_{o1}, Y_{o2} of (5.1) then need to be obtained by stepwise integration. With equal step-lengths an undesirable loss of numerical accuracy results if solutions are exponential in form. The difficulty is readily overcome by a change of variable from Y_{oj} to ln Y_{oj}, but a criterion is needed by which to determine in advance of the integration process whether or not such a form of solution is to be expected. A criterion which has been found suitable for treating the solution as exponential-like is that

$$|\Gamma_1^I| \gg |\Gamma_1^R| \tag{12.1}$$

for all values of x in $(0,x_E)$, where Γ_1 is defined by (6.9).

<u>Case (i)</u>: $|\Gamma_1^I| < |\Gamma_1^R|$ for at least one value of x in $(0,x_E)$

Y_{oj} is obtained by stepwise integration of (5.1) in the form of two first-order equations

$$\frac{dY_{oj}}{dx} = Y_{1j} \tag{12.2}$$

$$\frac{dY_{1j}}{dx} = -R\,Y_{oj} + 2\psi\,Y_{1j} \tag{12.3}$$

where Y_{1j} is an auxiliary variable. The integration is started with initial conditions

$$\begin{aligned} Y_{01}(0) &= (\bar{G})^{-\frac{1}{4}} & \frac{dY_{01}(0)}{dx} &= 0 \\ Y_{02}(0) &= 0 & \frac{dY_{02}(0)}{dx} &= (\bar{G})^{\frac{1}{4}} \end{aligned} \tag{12.4}$$

where $\bar{G}$ is arbitrarily taken to be the average value of G in $(0,x_E)$. From (5.3) and (12.4)

$$w_o(0) = 1 \tag{12.5}$$

<u>Case (ii)</u>: $|\Gamma_1^I| \gg |\Gamma_1^R|$ for all x in $(0,x_E)$

We write

$$X_{0j} = \ln Y_{0j} + K \qquad (12.6)$$

where K is a constant which it will be shown can be chosen such that (12.5) holds. An auxiliary variable X_{1j} is introduced by

$$X_{1j} = dX_{0j}/dx \qquad (12.7)$$

then (5.1) reduces to a Riccati's equation

$$\frac{dX_{1j}}{dx} = -R + 2\psi X_{1j} - X_{1j}^2 \qquad (12.8)$$

Initial conditions for Y_{01}, Y_{02} are taken to be

$$Y_{01}(0) = 2^{-\frac{1}{2}}(-\bar{G})^{-\frac{1}{4}} \qquad \frac{dY_{01}(0)}{dx} = -2^{-\frac{1}{2}}(-\bar{G})^{\frac{1}{4}} \qquad (12.9)$$

$$Y_{02}(0) = 2^{-\frac{1}{2}}(-\bar{G})^{-\frac{1}{4}} \qquad \frac{dY_{02}(0)}{dx} = 2^{-\frac{1}{2}}(-\bar{G})^{\frac{1}{4}} \qquad (12.10)$$

where $\bar{G}$ is arbitrarily taken to be the average of G in $(0, x_E)$. (12.9) and (12.10) are appropriate to solutions that initially decrease and increase respectively with exponential form from the same initial value and satisfy (12.5). By (12.6), (12.7) the corresponding conditions for X_{0j} are

$$X_{01}(0) = 0 \qquad \frac{dX_{01}(0)}{dx} = -(-\bar{G})^{\frac{1}{2}} \qquad (12.11)$$

$$X_{02}(0) = 0 \qquad \frac{dX_{02}(0)}{dx} = (-\bar{G})^{\frac{1}{2}} \qquad (12.12)$$

on choosing

$$K = \frac{1}{2} \ln\left[2(-\bar{G})^{\frac{1}{2}}\right] \qquad (12.13)$$

The computational procedure would then be to integrate (12.8) followed by integration of (12.7) for each of the initial conditions (12.11), (12.12) and then Y_{o1}, Y_{o2} would follow from (12.6) using (12.13).

When (12.8) was integrated with initial conditions (12.11), (12.12) for the case of G real and negative, corresponding to a non-dissipative atmosphere and solar diurnal modes with negative n, it was found that the solution with (12.11) after initially decreasing changed to one that increased with x. The reason for this behaviour may be seen by writing (12.8) as

$$\frac{d}{dx}(X_{1j}-\psi) = (-G) - (X_{1j}-\psi)^2 \qquad (12.14)$$

by (3.17) and (6.3). For a non-dissipative atmosphere, $\psi = 0$, and with initial condition (12.12), variations in G from $\bar{G}$ result in a solution $X_{12}(x) \approx \left[-G(x)\right]^{1/2}$ and $dX_{12}/dx \approx 0$. With initial condition (12.11) however the same variations in G from $\bar{G}$ result in a solution $X_{11}(x) \not\approx -\left[-G(x)\right]^{1/2}$ as there is no corresponding change of sign in dX_{11}/dx calculated from (12.14). For sufficiently large x, it was found that X_{o1}, X_{11} were approximately proportional to X_{o2}, X_{12} respectively. In principle the solutions obtained for Y_{o1}, Y_{o2} are valid, independent solutions; but a serious difficulty arises because as x increases they become dependent solutions to the accuracy of computation, and in subsequent calculations all significant figures are lost through the differencing of very large and nearly equal numbers.

The following procedure has enabled exponentially decreasing solutions to be obtained without loss of accuracy. The solution with initial conditions (12.12) is first obtained as described. Then (12.8) is integrated backwards from $x = x_E$ to $x = 0$ with the initial condition

$$X_{11}(x_E) = -X_{12}(x_E) \tag{12.15}$$

to obtain $X_{11}(x)$; and $X_{01}(x)$ is obtained from (12.7) by likewise integrating backwards from $x = x_E$ to $x = 0$ with the initial condition

$$X_{01}(x_E) = -X_{02}(x_E) \tag{12.16}$$

By (12.6), (12.7) we obtain from (5.3)

$$w_0(x_E) = \left\{\exp\left[X_{01}(x_E) + X_{02}(x_E) - 2K\right]\right\}\left[X_{12}(x_E) - X_{11}(x_E)\right] \tag{12.17}$$

Hence by (5.4), (12.15) and (12.16)

$$w_0(0) = 2X_{12}(x_E)\exp\left[-2K - 2\int_0^{x_E}\psi\,du\right] \tag{12.18}$$

In order that (12.5) should hold, we choose

$$K = \frac{1}{2}\ln\left[2X_{12}(x_E)\right] - \int_0^{x_E}\psi\,du \tag{12.19}$$

By (12.6) the required solutions are

$$Y_{0j}(x) = \left\{ \exp\left[X_{0j}(x) + \int_0^{x_E} \psi \, du\right]\right\} \Big/ \left[2 X_{12}(x_E)\right]^{\frac{1}{2}} \qquad (12.20)$$

Computing accuracy may be checked in the usual way by changing the step-length or by evaluating the Wronskian by (5.3) and comparing the value with that of $\exp\left[2\int_0^x \psi \, du\right]$ (Equ. 5.4 and 12.5). Integration step-lengths have been chosen according to the scheme in Table 1. With $\ell = \frac{1}{4}$ km, an accuracy of 1 in 10^6 has been maintained in the computations undertaken.

Table 1. Choice of step-length according to h, where ℓ is step-length selected for $0.5 \leqslant h^{-1/2} < 1.25\ (\mathrm{km})^{-1/2}$

$h < 0$: $(-h)^{-1/2}$ $(\mathrm{km})^{-1/2}$	0 - 1,	1 - 2,	> 2
step-length	$\frac{1}{2}\ell$	$\frac{1}{4}\ell$	$\frac{1}{8}\ell$

$h > 0$: $h^{-1/2}$ $(\mathrm{km})^{-1/2}$	0-0.5,	0.5-1.25,	1.25-2.5,	2.5-5,	> 5
step-length	$\frac{1}{2}\ell$	ℓ	$\frac{1}{2}\ell$	$\frac{1}{4}\ell$	$\frac{1}{8}\ell$

At the upper end of the interval $(0, x_E)$, the solutions Y_{01}, Y_{02} are required to approximate to the WKBJ form of

(6.12) in order to determine $C_{r_o 1}$ from (6.16). We define

$$Z_{r_1}(x) = \frac{i\,\frac{dY_{o1}(x)}{dx} + K_{r_1}(x)\,Y_{o1}(x)}{i\,\frac{dY_{o2}(x)}{dx} + K_{r_1}(x)\,Y_{o2}(x)} \quad (0 \leq x \leq x_E) \qquad (12.21)$$

where Y_{oj}, dY_{oj}/dx are obtained by numerical integration as described above and may not necessarily be WKBJ solutions. In (12.21), $r = r_o$, r_o being 1 or 2 according to the type of upper boundary condition, i.e. whether $K^I_{r_o}(x_E) < 0$ or $K^R_{r_o}(x_E)/h > 0$. If the values of $Z_{r_o 1}(x)$ approximate to a constant value for a range of values of x at the upper end of the interval $(0, x_E)$, then this constant value has been taken to be the quantity defined by (6.16) in which Y_{oj}, dY_{oj}/dx are WKBJ solutions.

Calculations have been undertaken for a non-dissipative atmosphere, i.e. one for which $a_T = \gamma = 0$ and (10.17) holds. Fig. 8 shows Y_{o1}, Y_{o2} at heights from 0 to 150 km for various values of h for which the Case (i) integration procedure applies. As h increases from 1 to 25 km, Y_{o1}, Y_{o2} change from an oscillatory to a mainly exponential form.

In choosing r_o we need to consider the following relations which derive from (6.9) and (7.9) on noting that for a non-dissipative atmosphere G, G_x are real

$$\begin{aligned} K_1^R = -K_2^R &= G^{\frac{1}{2}} \\ K_1^I = K_2^I &= G_x/4G \end{aligned} \qquad (G > 0) \qquad (12.22)$$

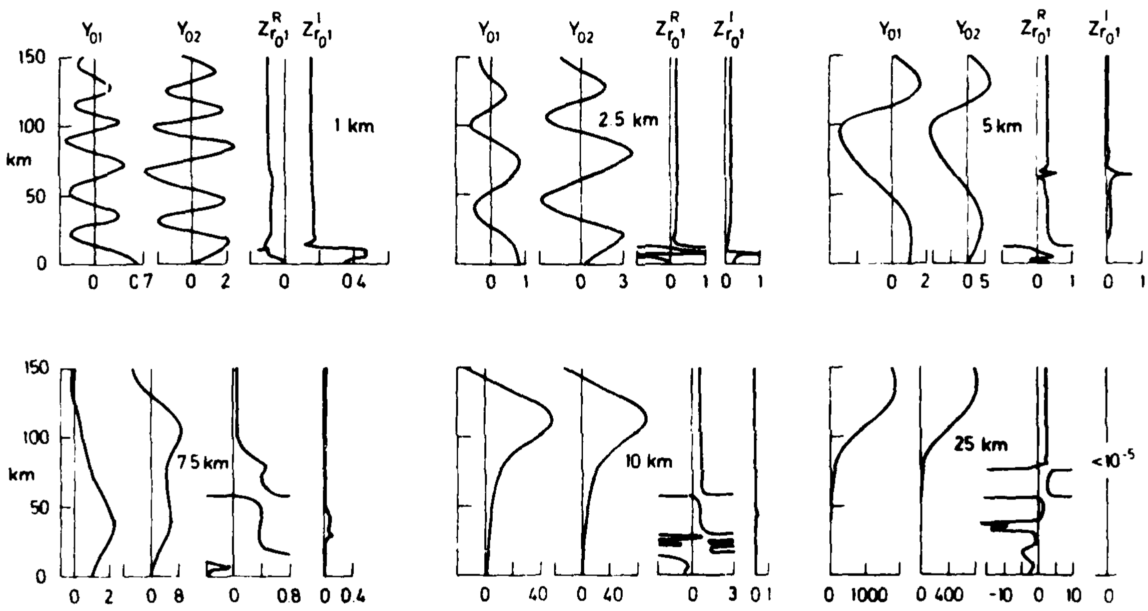

Fig. 8 Independent solutions Y_{01}, Y_{02} of the homogeneous equation (5.1). Values of h are shown on each set of curves. Case (i) integration procedure applies. $Z^R_{r_01}$, $Z^I_{r_01}$ are the real and imaginary parts of Z_{r_01}(Equ. 12.21) which determines C_{r_01} on approximating to a constant value with increasing height. Calculations are for the scale height of Fig. 1 and a non-dissipative atmosphere having $\gamma = 7/5$.

$$K_1^R = K_2^R = 0$$
$$K_1^I = (-G)^{\frac{1}{2}} + G_x/4G \qquad (G < 0) \qquad (12.23)$$
$$K_2^I = -(-G)^{\frac{1}{2}} + G_x/4G$$

where by (10.17)

$$G_x = \frac{H}{h}\left[\left(\kappa + \frac{dH}{dz}\right)\frac{dH}{dz} + H\frac{d^2H}{dz^2}\right] \qquad (12.24)$$

For all cases in Fig. 8 $G(x_E) > 0$, where x_E corresponds to 150 km altitude, and hence by (12.22) with $x = x_E$ the choice $r_o = 1$ complies with the upper boundary condition (iii) (b) of § 7, i.e. $K^R_{r_o}(x_E)/h > 0$, $K^R_{r_o'}(x_E)/h < 0$. Also it may be noted that condition (iii) (a) of § 7 does not hold as $K_1^I K_2^I \geqslant 0$ by (12.22).

Values of $Z_{r_o 1}$ calculated from (12.21) with $r_o = 1$ are plotted in Fig. 8 and above 100 km approximate to constant values which determine $C_{r_o 1}$ with an accuracy of the order of 10^{-3} in comparison with $C_{r_o 2} = 1$. The values obtained for $C_{r_o 1}$ depend on the arbitrarily chosen solutions Y_{o1}, Y_{o2} and have no direct physical significance. $C_{r_o 1}$ enters the analysis through (7.16) which is valid for any independent solutions Y_{o1}, Y_{o2} and the corresponding value of $C_{r_o 1}$.

Fig. 9 shows Y_{o1}, Y_{o2} for various values of h for which the Case (ii) integration procedure applies whereby one

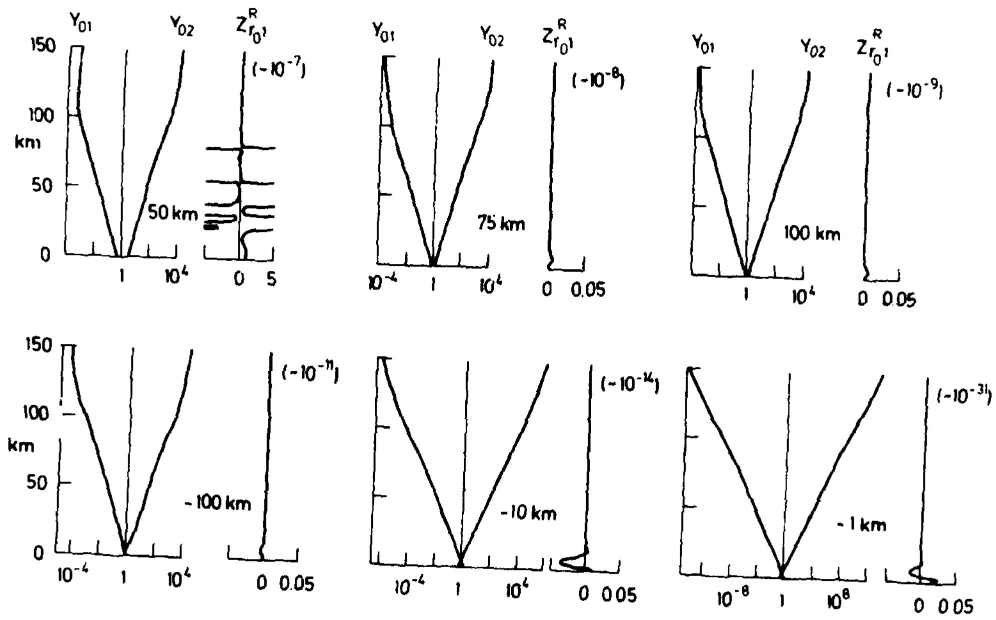

Fig. 9 As for Fig. 8 except that Case (ii) integration procedure applies and $Z^I_{r_0 1} \equiv 0$.

exponentially increasing solution and one exponentially decreasing solution are obtained. For h = 50 km, $G(x_E) > 0$ and $r_o = 1$ as for the cases in Fig. 8. For other values of h in Fig. 9 $G(x_E) < -\left[G_x(x_E)/4G(x_E)\right]^2 < 0$, and hence by (12.23) $K_1^I K_2^I < 0$ and the appropriate choice of r_o is $r_o = 2$ under condition (iii) (a) of § 7, i.e. $K_2^I < 0$ and $K_1^I > 0$.

For all cases in Fig. 9, $Z^I_{r_o 1} \equiv 0$ and $Z^R_{r_o 1}$ decreases at sufficiently great heights to very small values indicating that $C_{r_o 1} \doteq 0$. $C_{r_o 1}$ is however multiplied by Y_{o2} in the product $C_{r_o 1} \wedge Y_o$ and as Y_{o2} increases indefinitely it does not necessarily follow that the product is negligibly small. The orders of magnitude of $Z^R_{r_o 1}$ at 150 km are shown in Fig. 9 and when multiplied by Y_{o2} the product is seen to be negligibly small compared with unity. The same result may be established analytically using the relation

$$Y_{o1}\frac{dY_{o2}}{dx} - Y_{o2}\frac{dY_{o1}}{dx} = 1 \qquad (12.25)$$

which follows from (5.3), (5.4) and (12.5). Then by (6.16) and (12.25)

$$C_{r_o} \wedge Y_o = \left[i K_{r_o} Y_{o2} - dY_{o2}/dx\right]^{-1} \qquad (12.26)$$

If $r_o = 1$, as for the case h = 50 km, $G(x) > 0$ for values of x close to x_E and by (12.22) and (12.26)

$$\left| C_{r_0} \wedge Y_0 \right| = \left[G\, Y_{02}^2 + \left(\frac{dY_{02}}{dx} + \frac{G_x}{4G} Y_{02} \right)^2 \right]^{-\frac{1}{2}} \qquad (12.27)$$

Hence if Y_{02}, dY_{02}/dx are large compared with unity the right-hand side of (12.27) is correspondingly small compared with unity as $G > 0$. If $r_0 = 2$ as for the other values of h in Fig. 9, $G(x) < 0$ for values of x close to x_E and by (12.23) and (12.26)

$$\left| C_{r_0} \wedge Y_0 \right| = \left| K_1^I\, Y_{02} + \frac{dY_{02}}{dx} \right|^{-1} \qquad (12.28)$$

Hence for large Y_{02}, dY_{02}/dx (which are positive for the exponentially increasing solution) the right-hand side of (12.28) is correspondingly small compared with unity as $K_1^I > 0$.

13. Surface pressure oscillation due to a tidal potential

The variation of the surface pressure oscillation $F_\Omega(0)$ with the equivalent depth h has been extensively investigated for different profiles of scale height H in connection with the theory of atmospheric resonance. We follow earlier procedure (Wilkes, 1949; Jacchia & Kopal, 1952; Siebert, 1961; Butler & Small, 1963; Giwa, 1968) and take the ratio

$$\eta(r_0,h) = (P_\Omega / P_e)_{x_A=0} \tag{13.1}$$

where P_Ω is given by (10.13) and P_e is the dimensionless height-dependent function for the equilibrium tide.

For the solar semi-diurnal oscillation it is now recognized that the effect of thermal excitations dominates the gravitational tide, but for the lunar semi-diurnal oscillation this is clearly not the case and the evaluation of η for the appropriate value of h (= 7.07 km) has been of continuing interest. By direct numerical integrations of the classical tidal equation, results for η with h = 7.07 km have been shown to be extremely sensitive to the choice of basic atmospheric properties, namely the profiles of scale height and Newtonian cooling (Chapman & Lindzen, 1970). In descriptive terms, the sensitivities arise from the setting up of a dependence on multiple reflexions between horizontal surfaces at different heights which are critically dependent on basic atmospheric properties. When the zonal winds and latitudinal temperature gradients of a realistic atmosphere are introduced the reflecting surfaces are no longer horizontal and by integration of the non-classical equations multiple reflexions and resulting sensitivities are found to be largely removed (Lindzen & Hong, 1974). We shall nevertheless develop the analysis for η by classical theory in view of the earlier attention that it received, but for some range of values of h including h = 7.07 km it is recognized that the results are not realistic.

By definition the static equilibrium tide has zero horizontal components of velocity, and hence from (3.3) and (3.12) $Y^U = Y^V = Y^P = 0$. From (3.3) we then obtain

$$P_e = -e^{-x}\Omega\, a_o/H \tag{13.2}$$

By (10.13) with x_A = 0 and (13.2) with x = 0, we may write (13.1) by (10.15) and (12.5) as

$$\eta(r_o,k) = \left[N_w(0,0) + i\int_0^{x_S} N_w(0,u)\,s_\Omega(u)\,du\right]H(0)/k \tag{13.3}$$

where the very small dependence of $\Omega(x)$ on x has been neglected. By (8.4), we may then write (13.3) as

$$\eta(r_o,k) = \frac{C_{r_o}\wedge Y_{oo}}{C_{r_o}\wedge Y_o^w(0)}\;\frac{H(0)}{k} \tag{13.4}$$

where

$$Y_{oo_j} = Y_{oj}(0) + i\int_0^{x_S} Y_{oj}(u)\,s_\Omega(u)\,du \tag{13.5}$$

An infinite response therefore arises for values of h which satisfy

$$C_{r_o}\wedge Y_o^w(0) = 0 \tag{13.6}$$

Oscillations for which (13.6) holds are referred to as free oscillations.

We proceed on the assumption that the atmosphere is non-dissipative, i.e. that $a = \psi = 0$ and that (10.17), (12.22) to (12.24) hold. Then by (5.6) $s_{\Omega} = 0$, by (13.5) $Y_{ooj} = Y_{oj}(0)$ and (13.4) may be written by (3.10) and (5.11) as

$$\eta(r_0, \hbar) = \frac{[C_{r_0} \wedge Y_0(0)]\, H(0)/\hbar}{C_{r_0} \wedge \left[\left(\frac{H(0)}{\hbar} - \frac{1}{2}\right) Y_0(0) + \frac{dY_0(0)}{dx}\right]} \qquad (13.7)$$

When Case (i) integration procedure applies, we obtain on introducing (12.4) into (13.7)

$$[\eta(r_0, \hbar)]^{-1} = 1 - \left(\tfrac{1}{2} + C_{r_0 1}\, G^{\frac{1}{2}}\right) \hbar / H(0) \qquad (13.8)$$

When Case (ii) integration procedure applies such that $Y_{01}(x)$ has an exponentially decreasing form and $Y_{02}(x)$ has an exponentially increasing form, we have that $C_{r_0 1} = 0$. Hence (13.7) gives by (12.6) and (12.7)

$$[\eta(r_0, \hbar)]^{-1} = 1 - \left[\tfrac{1}{2} - X_{11}(0)\right] \hbar / H(0) \qquad (13.9)$$

In the special case when $x_L = 0$, i.e. when WKBJ solutions hold for all x, $G(x)$ which is real by (10.17) has the same sign at all heights otherwise (6.4) would be invalidated. Under Case (i) integration procedure we have that $G(x) > 0$, and from (12.22) it follows that the choice $r_o = 1$ complies with the upper boundary condition (iii) (b) of § 7, i.e. $K^R_{r_o}(x)/h > 0$, $K^R_{r_o'}(x)/h < 0$ (as $h > 0$ by (10.17) in which $\kappa + dH/dz > 0$ for a realistic atmosphere). (6.16) now holds for $x = 0$ and by (12.4) gives

$$C_{11} = -i\, K_2(0)/\, \bar{G}^{\frac{1}{2}} \tag{13.10}$$

Hence (13.8) becomes

$$\begin{aligned}[\eta(1,k)]^{-1} &= 1 - \left[\tfrac{1}{2} - i\,K_2(0)\right] k/H(0) \\ &= 1 - \left\{\tfrac{1}{2} + \frac{G_x(0)}{4G(0)} + i\left[G(0)\right]^{\frac{1}{2}}\right\} k/H(0)\end{aligned} \tag{13.11}$$

by (12.22). Under Case (ii) integration procedure we have $G < 0$ and by (12.23) it follows that the choice $r_o = 2$ complies with the upper boundary condition (iii) (a) of § 7, i.e. $K^I_{r_o}(x) < 0$, $K^I_{r_o'}(x) > 0$. As $C_{21} = 0$, it follows from (6.16) that $A_{21}(x) = 0$ and from (6.12) that

$$Y_{01}(x) = A_{11}(\xi)\exp\left[i\int_{\xi}^{x} K_1(u)\,du\right] \tag{13.12}$$

Hence

$$X_{11}(0) = i\,K_1(0) \tag{13.13}$$

and (13.9) becomes

$$[\eta(2,h)]^{-1} = 1 - \left[\tfrac{1}{2} - iK_1(0)\right] h/H(0)$$

$$= 1 - \left\{ \tfrac{1}{2} + \frac{G_x(0)}{4G(0)} + \left[-G(0)\right]^{\frac{1}{2}} \right\} h/H(0) \qquad (13.14)$$

by (12.23).

For an atmosphere of constant scale height H, $G_x = G_{xx} = 0$ by (12.24), and (13.11) and (13.14) give

$$[\eta(1,h)]^{-1} = 1 - \left[\tfrac{1}{2} + i\left(\frac{\kappa H}{h} - \frac{1}{4}\right)^{\frac{1}{2}}\right]\frac{h}{H} \qquad \left(0 < \frac{h}{H} < 4\kappa\right) \qquad (13.15)$$

$$[\eta(2,h)]^{-1} = 1 - \left[\tfrac{1}{2} + \left(\frac{1}{4} - \frac{\kappa H}{h}\right)^{\frac{1}{2}}\right]\frac{h}{H} \qquad \left(\frac{h}{H} < 0 \text{ or } \frac{h}{H} > 4\kappa\right) \qquad (13.16)$$

Fig. 10 shows (13.15) and (13.16) plotted on an Argand diagram for a range of values of h/H. The only physical quantity that enters into the calculation of this curve is γ, the ratio of specific heats for air, which is taken to be 7/5. The semicircular part of the plot has radius κ and L is the point corresponding to the leading lunar semi-diurnal mode with H = 7.6 km. At the origin

$$\frac{H}{h} - \frac{1}{2} = \left(\frac{1}{4} - \frac{\kappa H}{h}\right)^{\frac{1}{2}} \qquad (13.17)$$

by (13.16). By (3.5), equation (13.17) gives

$$h = \gamma H \qquad (13.18)$$

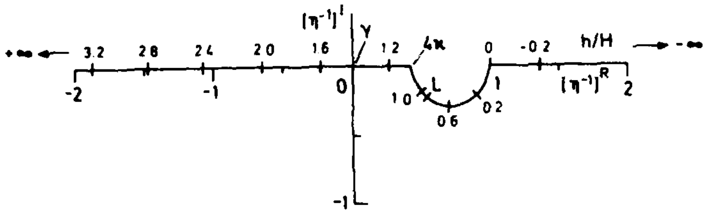

Fig. 10 Argand diagram of η^{-1} for a range of values of h/H, where scale height H is constant. $\gamma = 7/5$ and $4\kappa = 4(\gamma - 1)/\gamma = 8/7$.

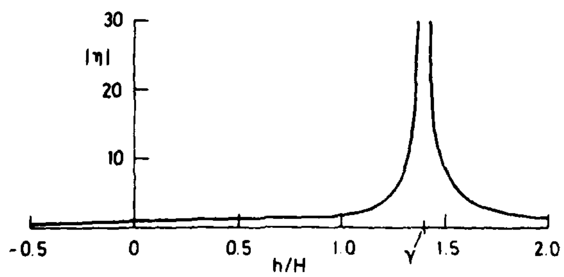

Fig. 11 $|\eta|$ against h/H, where η is shown in Fig. 10. $\gamma = 7/5$.

Fig. 11 shows the magnification $|\eta|$ as a function of h/H. The main feature of this curve is the free oscillation at $h/H = \gamma$: for H = 7.6 km the free oscillation arises when h = 10.64 km.

For a realistic scale height profile (Fig. 1) the evaluation of η by (13.8) or (13.9) generally requires integrations of the homogeneous equation (5.1) over a range $(0, x_E)$ for a value x_E such that WKBJ solutions are valid for $x \geqslant x_E$. For the scale height profile of Fig. 1, η^{-1} is shown in Fig. 12 for a range of values of h in km. (The curve between h = 7.3 and 9 km is omitted to avoid confusion with the part of the curve between h = 4 and 6 km which it follows closely). On comparing Figs. 10 and 12 (or Figs. 11 and 13), the change from a constant to a realistic scale height is seen to drastically effect the response for values of h from about 4 to 8 km including that of the leading lunar semi-diurnal mode at L (Figs. 10 and 12).

In Fig. 13 the magnification $|\eta|$ has a very sharp peak at h = 6.77 km with a magnification of 16.5. The larger peak at h = 10.3 km is similar to that for a constant scale height (Fig. 11) but the peak magnification, although very large, is no longer infinite. The two peaks of Fig. 13 have featured in previously published response curves

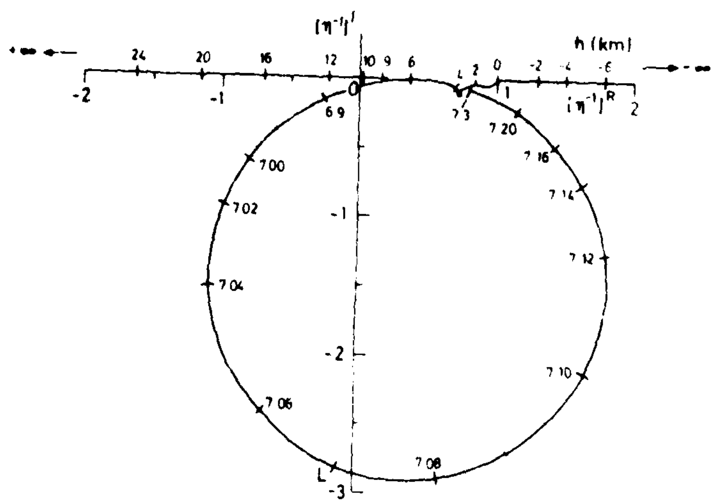

Fig. 12 Argand diagram of η^{-1} for a range of values of h. H is taken from Fig. 1. $\gamma = 7/5$.

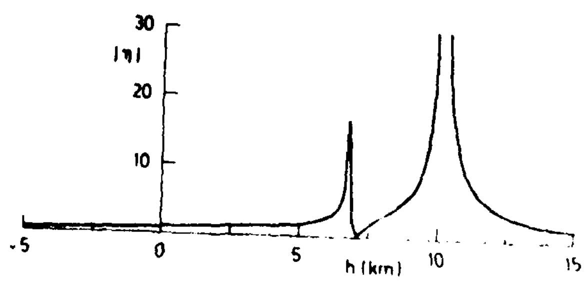

Fig. 13 $|\eta|$ against h, where η is shown in Fig. 12.

(Weekes & Wilkes, 1947; Jacchia & Kopal, 1952; Giwa, 1968; Hollingsworth, 1971). Giwa (1968) obtained a maximum at $h = 3.4$ km with a magnification of 1.7; and a third maximum is obtained in the present calculations at $h = 3.1$ km, but it is too small to be apparent in Fig. 13 although it is shown by the tight loop in Fig. 12. A still smaller maximum obtained by Giwa (1968) at $h = 1.9$ km would appear to correspond to the inflexion that is apparent in the curve in Fig. 12.

As multiple reflexions do not arise with a constant scale height a more realistic result for η^{-1} might be expected to be shown by Fig. 10 than Fig. 12, in spite of the scale height approximation involved. Using the equation of the locus shown in Fig. 10, Chapman & Lindzen (1970) pointed out that both the observed lunar semi-diurnal pressure amplitude of ≈ 70 μb and phase of $\approx 72^o$ are quite well represented by taking $H = 6.76$ km. The predicted phase can be read off directly from Fig. 10 as the angle between OL and the downward axis, where O is the origin. L is plotted in Fig. 10 for $H = 7.6$ km and a phase of 68^o is then obtained.

Associated with an externally applied tidal potential Ω_{oE} is an additional potential $k\Omega_{oE}$ arising from deformation of the Earth and a vertical velocity component at the Earth's solid surface of $-h'\dot{\Omega}_{oE}/g_o$, where h' and k are Love's numbers.

For ocean regions an additional vertical velocity component w_{oC} is introduced by the ocean tide and there is also an additional potential Ω_{oC} due to deformation of the ocean (Hollingsworth, 1971). If the individual excitations are taken to be additive we have

$$\begin{aligned} \Omega_o &= (1+k)\Omega_{oE} + \Omega_{oC} \\ w_o &= w_{oC} - h'\dot{\Omega}_{oE}/g_o \end{aligned} \qquad (13.19)$$

Then for a single mode by (2.21) and (2.22)

$$\Omega = (1+k)\Omega_E + \Omega_C \qquad (13.20)$$

$$W = W_C + ih'\Omega_E \qquad (13.21)$$

If the surface oscillations generated by (13.20) and (13.21) are denoted by $P_{o\Omega}$, P_{oW}, the oscillation recorded by a land-based barometer is

$$P_o = P_{o\Omega} + P_{oW} + h'\Omega_{oE}\,\rho_{oo}/g_o\,H(0) \qquad (13.22)$$

The third term on the right-hand side is the pressure oscillation due to the small vertical displacement of the barometer $-h'\Omega_{oE}/g_o$. For a single mode (13.22) becomes by (2.21) and (2.22)

$$P = P_\Omega + P_W + h'\Omega_E\,a_o/H(0) \qquad (13.23)$$

Hence by (10.2), (10.12) and (10.13) with $x_A = 0$

$$\mathcal{P} = \frac{a_o L_p(0,0)}{i\,H(0)} W_c - \frac{a_o}{\hbar}\Big\{ N_w(0,0)\big[(1+k-h')\Omega_E + \Omega_c\big] + i\int_0^{x_s} N_\Omega(0,u)\,\Omega(u)\,du \Big\} \tag{13.24}$$

If the contributions of the oceans and atmospheric dissipation are neglected we have

$$\mathcal{P} = -(1+k-h')\,\Omega_E\, N_w(0,0)\, a_o/\hbar \tag{13.25}$$

(13.25) shows that the elasticity of the Earth has the effect of multiplying Ω_E and hence the resulting tidal pressure amplitude by (1+k-h'), i.e. by about 0.70, which is the result given by Hollingsworth (1971).

14. Discussion

Atmospheric tidal theory has been developed and applied over many decades, but a comprehensive analytical treatment along the lines of that of Butler & Small (1963), which essentially involves the forming of Green's functions, has been lacking.

A difficulty that soon arises in any such analysis is that of manipulating rather cumbersome equations. Close attention has therefore been given here to the choice of notation and formulation: in particular it seemed worthwhile introducing the symbol $\wedge$ for the cross-product that frequently occurs. A feature of the analysis has been the systematic formulation of results for different atmospheric variables, i.e. wind components, temperature and pressure. Progress in this direction was helped by the introduction of the Y^Q notation (Equ. 3.3) and the derivation of (5.10). Another feature of the analysis has been the retention of the arbitrary constants Ξ_{o1}, Ξ_{o2} until they could be simultaneously eliminated by the introduction of two boundary conditions to give the determinant (8.1) and thence the general relations of § 8. The upper boundary condition follows earlier accounts by adopting the radiation condition, but its application in terms of WKBJ solutions is less restrictive as the actual structure of the atmospheric 'top' does not need to be specified. At the lower boundary the vertical component of velocity is retained as the usual assumption of a zero value holds only for a surface-air interface that is rigid and horizontal. Undulations of the terrain interact with the primary atmospheric oscillations and set up new oscillations for which vertical velocity components at the surface are not necessarily zero. Surface tidal motions are another source of excitation at the lower boundary to which the present results are applicable in terms of vertical motion.

Application of the derived formulae to quantitatively defined sources of excitation has not been undertaken within the scope of this paper. The analysis has however been developed for particular heights: the first case at the lower boundary (§ 10) leads to the evaluation of surface oscillation weighting functions W_p for a range of solar diurnal and semi-diurnal modes (Equ. 10.16, Figs. 2 to 4). The second case is for heights above the region of excitation and leads to the evaluation of the thermal response weighting function W_t for the same modes (Equ. 11.8, Figs. 5 to 7). W_p, W_t have previously been presented graphically for a selection of modes and provide a useful means of understanding the relationship between the vertical structure of a thermal excitation and its resulting atmospheric response (Groves, 1975, 1976, 1977).

No attempt has been made to define the limitations of classical theory in its application to the real atmosphere, but reference is made in § 13 to the case of excitation by the leading lunar semi-diurnal mode of tidal potential for which classical results may be unrealistic. The treatment of the surface pressure oscillation due to a tidal potential (§ 13) has been of long-standing interest and provides an example of the application of (10.13): the results obtained are gratifyingly in close agreement with previous accounts.

Acknowledgements - Sponsorship has been provided for this work by the Air Force Geophysics Laboratory (AFGL), United States Air Force under Grant No. AFOSR - 77-3224. The assistance of Mr. Jonathan D. Groves and Mr. Andrew Wilson with computations on the GEC 4082 machines (EUCLID) at University College London is very gratefully acknowledged.

List of symbols

	Reference		Reference
a_o	(2.13)	p_o, p_{oo}	(1.1)
a_j, a'_j	(6.5)	p_n^s	(1.7)
a_T	(3.6)	r'	(6.11)
f	(2.13)	r_o	(7.11)
g	(1.4)	s_o	(2.9)
g_o	(2.13)	s_J, s_Ω	(5.6)
h	(1.2)	t	(2.2)
h'	(13.19)	t'	(2.10)
h_n^s	(1.4)	w_o	(5.3)
k	(13.19)	x	(1.1)
ℓ	Table 1	x_o	(5.4)
ℓ_Q	(2.2)	x_A	(5.5)
ℓ_E, ℓ_J, ℓ_L,		x_B	(5.14)
ℓ_P, ℓ_T, ℓ_U,		x_E	(12.15)
ℓ_V, ℓ_W, ℓ_Ω	(2.22)	x_L	(7.2)

	Reference
x_S	(7.1)
y	(1.2)
y_1, y_2	(1.5)
y_n^s	(1.3)
z	(1.1)
A_r, A_{rj}	(6.10)
C_r, C_{rj}	(6.16)
D_o, D_{oj}	(5.7)
E	(4.5)
$\overline{E}$	(4.1)
E_r, E'	(6.22)
F	(3.18)
G	(6.2)
$\overline{G}$	(12.4)
G_n^s	(1.3)
H	(1.1)
I_r	(6.6)
I_n^s	(1.5)
J	(2.21)
J_o	diabatic heating rate per unit mass of atmosphere
J_n^s	(1.4)

	Reference
K	(12.6), (12.14), (12.19)
K_r	(6.8)
L	(2.21), (3.5)
L_o	rate of decrease of temperature, (3.7)
L_Q $(Q=P,T,U,V,W)$	(9.16)
M_Q	(8.2)
$M_{QQ'}$, $(Q=P,T,U,V,W)$ $(Q'=J,\Omega)$	(9.15)
N_Q $(Q=P,T,U,V,W)$	(8.2)
N_Q $(Q=J,\Omega)$	(10.15)
P	(2.21)
P_o	pressure perturbation
P_e	(13.1)
P_r	(6.17)
$P_{o\Omega}$, P_{oW}	(13.22)
Q	(2.21)
Q_o	[illegible]
Q_o'	(2.20)
Q_c	(2.4)
Q_J, Q_W, Q_Ω $(Q=P,T,U,V,W)$	([illegible].1)
Q^s	(2.6)
Q_n^s	(2.11)

	Reference		Reference
Q_C^s, Q_S^s	(2.5)	Y_{1j}	(12.2)
R	(3.17)	Y_{oo}, Y_{ooj}	(13.4)
S	(5.5)	Y_a, Y_{aj}	(6.1)
T	(2.21)	Y^P, Y^T, Y^U, Y^V, Y^W	(3.3)
T_o	temperature perturbation	Y^Q	(3.9)
U	(2.21)	Y_o^Q, Y_{oj}^Q, (Q= P,T,U,V,W)	(5.10), ([illegible].1[illegible])
U_o	perturbation of the eastward wind component	ω_{rj}	(1[illegible].21)
V	(2.21)		
V_o	perturbation of the northward wind component	α	(1.5)
		β	(1.[illegible])
W	(2.21)	γ	(1.[illegible])
W_o	perturbation of the vertical wind component	η	([illegible].1)
		θ	([illegible].1)
W_{oC}	(13.19)	κ	([illegible])
W_p	(10.16)	λ	([illegible].4)
W_t	(11.8)	μ	([illegible].1)
W_C	(13.21)	ξ	(6.[illegible])
X_{oj}	(12.6)	ξ'	(6.14)
X_{1j}	(12.7)	ρ_o	(4.[illegible])
Y	(3.2), (5.2)	σ	(1.[illegible]), ([illegible])
Y'	(5.2)	ϕ	(2.[illegible])
Y_o, Y_{oj}	(5.1)		

	Reference
Ψ	(3.16)
ω_o	(2.13)
Γ_1, Γ_2	(6.8)
Θ	(2.21)
Θ_U	(2.15), (2.21)
Θ_V	(2.16), (2.21)
$\Theta_n^{\wedge}$	(2.11)
Ξ_o, Ξ_{oj}	(5.2)
Φ	(11.1)
Ω	(2.21)
Ω_o	potential of applied force per unit mass of atmosphere
Ω_{oC}, Ω_{oE}	(13.19)
Ω_C, Ω_E	(13.20)
$\mathcal{B}$	(3.14)

	Reference
$\mathcal{F}_P, \mathcal{F}_T, \mathcal{F}_U,$	
$\mathcal{F}_V, \mathcal{F}_W$	(3.10)
$\mathcal{F}_Q$	(3.[illegible]

Superscripts

R	real part
I	imaginary part
*	complex conjugate

Subscripts

j	= 1, 2 refers to the two independent solutions of (5.1)
r	= 1, 2 refers to the two WKBJ exponential forms
x	derivative with respect to x

Symbol

$\wedge$	$a \wedge b = a_1 b_2 - a_2 b_1$

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